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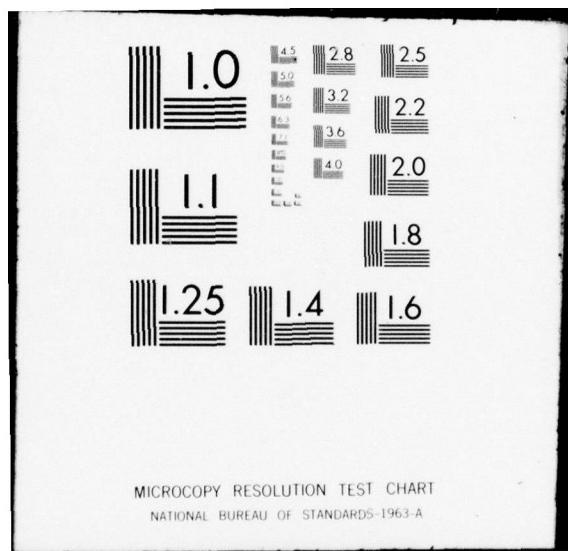
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Downtime Modeling and Optimal Replacement
of Deteriorating Parts

10 by C. L. Hsu, L. Shaw ■ S.G. Tyan

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ABSTRACT

→ In the area of application of probabilistic and stochastic modeling sequences of random variables evolving over time are usually assumed to be sequences of independent random variables or Markov sequences. Here we introduce and apply a multivariate exponential distribution which may describe Markov or non-Markov sequences. The present work has examined one particular class of multivariate exponential distributions which preserve Markov sequence properties for both modeling of downtime distributions and modeling of stages of component deterioration. In downtime modeling, we study the distribution of the sum of several dependent random variables and compare the result with the distribution of a sum of independent variables as well as with the lognormal distribution. In deterioration modeling, we consider part replacement rules based on observation of the state of the part's quality and on specified reward structures. We identify the rate of deterioration by examining how long the component stays in each state and use dynamic programming to set up recursive optimization equations such that the expected reward per unit time is maximized. Sufficient conditions are given under which the optimum replacement rule has a very simple structure.

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by

C. L. Hsu and L. Shaw and S. G. Tyan

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A B S T R A C T

In the area of application of probabilistic and stochastic modeling, sequences of random variables evolving over time are usually assumed to be sequences of independent random variables or Markov sequences. Here we introduce and apply a multivariate "exponential" distribution which may describe Markov or non-Markov sequences. The present work has examined one particular class of multivariate exponential distributions which preserve Markov sequence properties for both modeling of downtime distributions and modeling of stages of component deterioration. In downtime modeling, we study the distribution of the sum of several dependent random variables and compare the result with the distribution of a sum of independent variables as well as with the lognormal distribution. In deterioration modeling, we consider part replacement rules based on observation of the state of the part's quality and on specified reward structures. We identify the rate of deterioration by examining how long the component stays in each state and use dynamic programming to set up recursive optimization equations such that the expected reward per unit time is maximized. Sufficient conditions are given under which the optimum replacement rule has a very simple structure.

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CHAPTER I

INTRODUCTION

Two important considerations in the design and the evaluation of systems are their capability to return to normal operation after severe deterioration and that after occurrence of failures. This study develops and analyzes stochastic models of systems whose components can be replaced without occurrence of failure and can be repaired when the failure does occur. The analysis focuses on a model in which system states are distinct at any time: the system is operating (up) at one of the different grades of performance or it has failed (down) and is at one of the different stages of repair.

With the advent of modern technology, system states can be observed with a non-destructive testing (e.g., acoustic wave, infrared, laser, x-ray, . . . , etc.) without inducing performance degradation or interrupting repair. Also the data is easy to analyze on a special purpose minicomputer, a microcomputer, or even a microprocessor.

One of the great challenges which engineering faces is the development of large and complex systems for both commercial and military applications. Outstanding examples of such systems are: nuclear power plants, switching machines of telephone office, air traffic control systems, radar detection and warning systems, space vehicles, aircraft, communication networks, real-time computer, and medical instruments. Failures of these complex systems might be fatal to human life. Replacement of deteriorating parts will reduce the probability of failure. Once a failure has occurred, repairability is an important consideration for restoring the system back to normal as soon as possible.

The design of maintenance policies for maintainable and repairable systems makes use of information about probability distributions of component lifetimes and downtimes. Such designs are facilitated if these distributions have simple analytic forms.

Downtime distributions have been frequently modeled as lognormal, Weibull, or Erlang, in form [7]. These distributions are all skewed and corresponding to non-negative random variables like downtimes. Here we consider one more family of distributions which has some physical motivation.

Since a downtime interval is often the sum of subsidiary intervals (for failure isolation, component removal, repair, reassembly, alignment, etc.) it seems reasonable to think of the downtime x_n as the sum of subsidiary time intervals:

$$x_n = \sum_{i=1}^n r_i \quad . \quad (1.1)$$

The subscript of x_n reminds us of the number of summands. Several distributions are possible for the individual r_i , but here we consider exponential distributions which are the simplest and are also widely used to represent random times between events. Clearly, once the distribution of such an x_n has been characterized, the assumption that r_i is exponentially distributed can be weakened to allow an x_n -type distribution for any r_i . For example, the repair time interval may not be characterized by an exponential distribution. However, by using the above argument, we can find a suitable n for an x_n -type distribution for the repair time.

It is well known that the downtime x_n will have an Erlang distribution if the r_i are independent exponential random variables with identical mean values. Muth [22] has considered the approximation of Weibull and lognormal distributions by x_n in which the r_i are independent exponential variables but with possibly different mean values. Here we generalize his work to allow dependence among the r_i -- a reasonable situation if the variables represent related steps in a sequence of downtime operations or in a sequence of deterioration levels.

Several types of multivariate exponential distributions have been proposed for various reliability applications [2]. The multivariate exponential distribution discussed most is the Marshall and Olkin's shock model. Marshall and Olkin's motivation was mainly for a system with parallel time intervals which happen to start at the same time (Figure 1(a)).

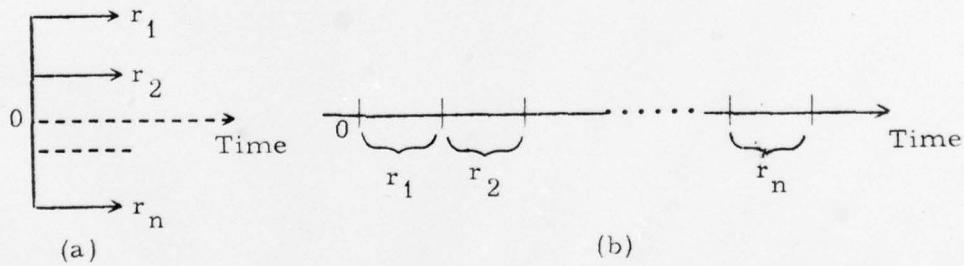


Figure 1:
Parallel and Series Random Time Intervals .

For our purpose we would like to have a sequence of random time intervals that happen one by one as in Figure 1(b).

The present work has examined one particular class of multivariate exponential distribution for both modeling of downtimes and modeling of component deterioration.

Chapter II discusses several forms of multivariate distributions which have exponential marginal distributions. The one based on the sums of the squares of normal variables is selected because of its analytical simplicity.

Chapter III gives examples of the distributions of the sums of n dependent exponential variables. We show that the introduction of dependence among r_i (with possibly unequal means) does not broaden the class of x_n distributions over that which results from independent r_i . That is, the sum of n dependent exponential variables has a distribution identical to that of the sum of n other independent exponential variables. Several conclusions are on the approximation of lognormal variables by sums of exponential ones, along with other possible extensions.

Chapter IV examines maintenance policies for systems in which the degree of deterioration can be observed continuously. A Markov sequence model is developed where the holding times in the various states are multivariate exponentially distributed. We assume that the functioning rewards during the system's lifetime decrease with the increasing deterioration. Additional dependence properties of multivariate exponential variables are developed for the optimization studies. The dependence relations seem to be consistent with what one expects of a general system in the real world. Optimal replacement rules which maximize the expected reward per unit time are obtained by using the dynamic programming method. Sufficient conditions which simplify the optimal decision rules are given in Appendix I. Chapter V summarizes the results and makes suggestions for further research.

Part of Chapter II and III have already appeared in a previous report by Hsu and Shaw [12].

CHAPTER II

MULTIVARIATE EXPONENTIAL DISTRIBUTIONS

The multinormal distribution has been studied far more extensively than any other multivariate distribution. Indeed, its position of pre-eminence among multivariate continuous distributions is more marked than that of the normal among univariate continuous distributions. However, in practical engineering problems there have been signs that the need for multivariate exponential distributions is becoming recognized. Several forms of multivariate exponential distributions have been introduced. The one based on the sums of squares of multinormal variables is selected here, because of its analytical simplicity through transformation to well developed multinormal distributions.

This chapter reviews several popular multivariate exponential distributions and develops properties of the one to be used in later chapters. Some important dependence properties will be developed in Chapter IV.

2.1 Review of Literature

It is well known that a multivariate distribution is not uniquely specified by its marginal distributions [2]. For example, while the bivariate normal distribution has nice analytical properties, it is not the only one with normal marginals. This multiplicity will be demonstrated for the case of exponential marginals by considering a few possible bivariate densities.

Gumbel [9] considered several bivariate exponential densities. The first $F_1(r_1, r_2)$ is based on the following general formula for combining marginal distributions:

$$F(x, y) = F_X(x) F_Y(y) \{ 1 + \alpha [1 - F_X(x)] [1 - F_Y(y)] \}, \quad (2.1)$$

$$|\alpha| \leq 1.$$

The marginal distributions of X and Y are each exponential. When applied to exponential variables with unit mean values this produces the distribution function:

$$F_1(r_1, r_2) = (1 - e^{-r_1})(1 - e^{-r_2})(1 + \alpha e^{-r_1 - r_2}), \quad (2.2)$$

$$r_1, r_2 \geq 0,$$

$$|\alpha| \leq 1,$$

and the density function:

$$f_1(r_1, r_2) = e^{-r_1 - r_2} [1 + \alpha (2e^{-r_1 - 1}) (2e^{-r_2 - 1})]. \quad (2.3)$$

In this model, $\alpha = 0$ corresponds to independence of r_1 and r_2 , and it can be shown that the correlation coefficient ρ_{r_1, r_2} is:

$$\rho_{r_1, r_2} = \alpha / 4, \quad (2.4)$$

with its magnitude limited to be less than or equal to $1/4$. Another model of Gumbel's is defined by the joint distribution function:

$$F_2(r_1, r_2) = 1 - e^{-r_1} - e^{-r_2} + e^{-r_1 - r_2 - \theta r_1 r_2}, \quad (2.5)$$

$$0 \leq \theta \leq 1, \quad r_1, r_2 \geq 0,$$

and the corresponding density:

$$f_2(r_1, r_2) = e^{-r_1 - r_2 - \theta r_1 r_2} [(1 + \theta r_1)(1 + \theta r_2) - \theta]. \quad (2.6)$$

Here $\theta = 0$ corresponds to independence, and:

$$\rho_{r_1, r_2} = -\theta^{-1} e^{+\theta^{-1}/\theta} E_i(\theta^{-1})^{-1},$$

where:

$$E_i(\theta^{-1}) = \int_{\theta^{-1}}^{\infty} e^{-Z} Z^{-1} dZ.$$

The univariate exponential distribution derives considerable importance from its role as the distribution of waiting time in a Poisson process. It is natural to inquire whether a similar relationship exists between some bivariate exponential distribution and the waiting times in a suitably defined two-dimensional Poisson process. One such possibility, investigated by Marshall and Olkin [2] and later generalized by them [21], will now be described.

This distribution can be thought of as a result of fatal shocks occurring from three independent Poisson sources with rates λ_1, λ_2 , and λ_{12} . Component 1 with lifetime r_1 is killed by events of the first or third variety, and r_2 is determined by events of the second or third type. We can define the reliability, which is a probability of survival as:

$$R(r_1, r_2) = P[R_1 > r_1, R_2 > r_2] \\ = e^{-\lambda_1 r_1 - \lambda_2 r_2 - \lambda_{12} \max(r_1, r_2)}, \quad (2.7)$$

or :

$$F_3(r_1, r_2) = \frac{e^{-(\lambda_1 + \lambda_{12})r_1} - e^{-r_2(\lambda_2 + \lambda_{12})}}{1 + e^{-r_1 \lambda_1 - r_2 \lambda_2 - \lambda_{12} \max(r_1, r_2)}}. \quad (2.8)$$

Here the correlation coefficient is :

$$\rho_{r_1, r_2} = \lambda_{12}/(\lambda_1 + \lambda_3 + \lambda_{12}) \quad (2.9)$$

which ranges between zero and one.

Barlow and Proschan [2] point out that Eq. (2.8) is the unique bivariate exponential distribution with the zero-memory property: the joint survival probability of a pair of components each of age t is the same for all t (e.g., the same as if both were new). This zero memory is quite desirable when modeling joint lifetimes of components in a system. In the present context of r_i representing durations of a sequence of related events, this special property seems inessential.

Kibble [18] considered a bivariate exponential density of the form:

$$f_4(r_1, r_2; \rho^2) = \frac{1}{4\sigma^4(1-\rho^2)} \left[e^{-\frac{r_1+r_2}{2\sigma^2(1-\rho^2)}} \right] I_0 \left[\frac{\sqrt{\rho^2 r_1 r_2}}{\sigma^2(1-\rho^2)} \right], \quad (2.10)$$

in which $I_0(\cdot)$ is a modified Bessel function of order zero defined as:

$$I_0(z) = \frac{1}{2\pi} \int_0^{2\pi} e^{z \cos \varphi} d\varphi = \sum_{k=0}^{\infty} \frac{(z/2)^{2k}}{(k!)^2}. \quad (2.11)$$

A detailed study of this distribution has been given by Nagao and Kadoya [23].

2.2 Multivariate Exponential Distribution

We will derive Equation (2.10) in an alternate, more natural way for use below. The same approach has also been taken by other authors, for example, Moran and Vere - Jones [32].

The density in (2.10) applies when we view r_1 and r_2 as being generated from correlated normal variables. It is well known that if w_1 and z_1 are independent, zero mean, equal variance (σ^2) normal variables then r_1 defined as :

$$r_1 = w_1^2 + z_1^2 , \quad (2.12)$$

has an exponential distribution with mean :

$$E(r_1) = 2\sigma^2 . \quad (2.13)$$

Now if (w_1, w_2) and (z_1, z_2) are two independent pairs of normal variables, but with:

$$\text{cov}(w_i, w_j) = \text{cov}(z_i, z_j) \quad i = 1, 2; \quad j = 1, 2 , \quad (2.14)$$

then r_1 and r_2 defined by :

$$r_i = w_i^2 + z_i^2 \quad i = 1, 2 , \quad (2.15)$$

will be dependent exponential variables.

Equation (2.10) can be derived from this reasoning when all four variables (w_1, w_2, z_1, z_2) have equal variances σ^2 . In that case :

$$f(w_1, w_2, z_1, z_2) = \frac{1}{(2\pi)^2 (1-\rho^2) \sigma^4} \exp \left[-\frac{w_1^2 - 2\rho w_1 w_2 + w_2^2 + z_1^2 - 2\rho z_1 z_2 + z_2^2}{2(1-\rho^2)\sigma^2} \right] , \quad (2.16)$$

and :

$$F_4(r_1, r_2) = \iiint f(w_1, w_2, z_1, z_2) dw_1 dw_2 dz_1 dz_2 ,$$

$$\left\{ \begin{array}{l} w_1^2 + z_1^2 \leq r_1 \\ w_2^2 + z_2^2 \leq r_2 \end{array} \right\} . \quad (2.17)$$

Introduction of polar coordinates in w_1, z_1 and w_2, z_2 planes:

$$w_1 = \gamma_1 \cos \theta_1 , \quad w_2 = \gamma_2 \cos \theta_2 ,$$

$$z_1 = \gamma_1 \sin \theta_1 , \quad z_2 = \gamma_2 \sin \theta_2 ,$$

reduces the F_4 integral to the form:

$$F_4(r_1, r_2) = \frac{1}{4\pi^2(1-\rho^2)\sigma^4} \int_{\gamma_1=0}^{\sqrt{r_1}} \int_{\gamma_2=0}^{\sqrt{r_2}} \int_{\theta_1=0}^{2\pi} \int_{\theta_2=0}^{2\pi} e^{-[\gamma_1^2 + \gamma_2^2 - 2\rho\gamma_1\gamma_2 \cos(\theta_1 - \theta_2)/2\sigma^2(1-\rho^2)]} \cdot \gamma_1 \gamma_2 s \gamma_1 d\gamma_1 d\gamma_2 d\theta_1 d\theta_2 \quad (2.18)$$

Substitution of $\varphi = \theta_1 - \theta_2$ for the θ_1 integration produces a periodic integrand, independent of θ_2 . Thus the θ -integrals become:

$$\begin{aligned} \int_{\theta_1=0}^{2\pi} \int_{\theta_2=0}^{2\pi} &= 2\pi \int_0^{2\pi} e^{\rho\gamma_1\gamma_2 \cos \varphi / \sigma^2(1-\rho^2)} d\varphi \\ &= 4\pi^2 I_0 \left[\frac{|\rho| \gamma_1 \gamma_2}{\sigma^2(1-\rho^2)} \right] . \end{aligned} \quad (2.19)$$

Finally, substitution of (2.19) into (2.18) and differentiations with respect to r_1 and r_2 produces (2.10). It turns out that ρ_{r_1, r_2} is just the square (ρ^2) of the correlation of the underlying normal variables, with a range between zero and one.

The Bessel function form for this bivariate exponential based on normal variates may not appear to be very felicitous. However, this kind of distribution will be convenient when we concentrate in the next chapter on the sum of dependent exponential variables, as in (1.1).

For the n-dimensional version of this class of distributions, we consider zero mean, normal n-vectors \underline{w} and \underline{z} each with the same covariance matrix (which is positively definite and symmetric):

$$E[\underline{w} \underline{w}'] = E[\underline{z} \underline{z}'] = \Gamma. \quad (2.20)$$

In this way, for each i , w_i and z_i will have the same variances so the sum of their squares will be an exponential random variable r_i . We do allow r_i and r_j to have unequal means, contrary to the special case in (2.10). Using the underlying normal distributions it is easy to show that the r_i have means:

$$E[r_i] = 2\gamma_{ii}, \quad (2.21)$$

and correlation coefficients:

$$\rho_{r_i, r_j} = (\gamma_{ij})^2 / (\gamma_{ii}\gamma_{jj}) \geq 0, \quad (2.22)$$

which can take on values between 0 and 1.

The general approach to calculation of the multivariate n-distribution is through integration of the normal density $f(w_1, w_2, \dots, w_n, z_1, z_2, \dots, z_n)$ over appropriate regions.

A multivariate case of interest corresponds to the covariance matrix:

$$\gamma_{ij} = \sigma_i \sigma_j \rho^{|i-j|}, \quad (2.23)$$

in which correlation between r_i and r_j falls off exponentially in relation to the separation between their indices i and j . The corresponding joint density has a nice structure, revealed by its trivariate form:

$$f(r_1, r_2, r_3) = \frac{1}{8\sigma_1^2 \sigma_2^2 \sigma_3^2 (1-\rho^2)^2} e^{-\left[\left(\frac{r_1}{\sigma_1^2} + \frac{r_2(1+\rho^2)}{\sigma_2^2} + \frac{r_3}{\sigma_3^2} \right) / 2(1-\rho^2) \right]} \\ \cdot I_0 \left[\frac{\sqrt{r_1 r_2 \rho^2}}{\sigma_1 \sigma_2 (1-\rho^2)} \right] \cdot I_0 \left[\frac{\sqrt{r_2 r_3 \rho^2}}{\sigma_2 \sigma_3 (1-\rho^2)} \right]. \quad (2.24)$$

This expression generalizes in an obvious way to the n -variate form:

$$f(r_1, r_2, \dots, r_n) = [(1-\rho^2)^{n-1} 2^n \prod_{i=1}^n \sigma_i^2]^{-1} \prod_{i=1}^{n-1} I_0 \left(\frac{\sqrt{r_i r_{i+1} \rho^2}}{\sigma_i \sigma_{i+1} (1-\rho^2)} \right) \\ \cdot \exp \left[-\frac{1}{2(1-\rho^2)} \left(\frac{r_1}{\sigma_1^2} + \sum_{i=2}^{n-1} \frac{r_i}{\sigma_i^2} (1+\rho^2) + \frac{r_n}{\sigma_n^2} \right) \right] \quad (n > 2). \quad (2.25)$$

The special case of Eq. (2.23) describes a Markovian normal sequence w_1, w_2, \dots, w_n with stationary correlation coefficients. The sequence r_1, r_2, \dots, r_n with a joint density of (2.25) can be easily seen to be Markovian also, a fact also established in Griffiths [13].

CHAPTER III

DOWNTIME MODELING

As mentioned in Chapter I, we want to represent a downtime as the sum of several, possibly dependent, subsidiary times :

$$x_n = \sum_{i=1}^n r_i \quad . \quad (3.1)$$

We will examine several possibilities for r_i distributions, beginning with a review of Muth's work [22] for the case of independent, exponentially distributed r_i with unequal mean values. Then we will generalize his model to allow dependence among the r_i , seeing that the family of possible f_{x_n} is not expanded in this way, but that it allows for convenient study of the case of large n and possible convergence to a log-normal density.

3.1 Review of Muth's Results

In Appendix B of Muth's thesis [22], he investigated the properties of the class of probability distributions belonging to the family of Erlang distributions, which are generated as the distribution of a sum of independent and exponentially distributed random variables. Specifically, he compared these distributions with the gamma, lognormal, and Weibull distributions, which are well known examples of two-parameter unimodal distributions and which have been used as models for repair times.

He considered the random variable $Y(n)$ which is the sum of n independent random variables X_i :

$$Y(n) = X_1 + X_2 + \dots + X_n , \quad (3.2)$$

where each X_i is exponentially distributed with mean value m_i . The random variable $Y(n)$ has a distribution in which n parameters, the values of m_1 to m_n , must be specified, and where n itself is a variable. We use the following notations and definitions:

$$E[Y(n)] = m, \quad \text{Var}[Y(n)] = \sigma^2, \quad (3.3)$$

$$\gamma_1 \text{ (skewness)} = \frac{E[\{Y(n)-m\}^3]}{\sigma^3}, \quad (3.4)$$

$$\gamma_2 \text{ (kurtosis)} = \frac{E[\{Y(n)-m\}^4]}{\sigma^4} - 3. \quad (3.5)$$

In order to compare the probability law of $Y(n)$ with that of a random variable Z , having a two parameter distribution, he set their means and variances equal:

$$E[Y(n)] = E[Z], \quad (3.6)$$

$$\text{Var}[Y(n)] = \text{Var}[Z], \quad (3.7)$$

and compared the higher moments of both probability laws. He restricted the comparison to the third and fourth moments. The underlying idea is that the approximation of one probability law by another becomes closer as one successively matches higher moments, and that matching the third moment, together with the constraints (3.6) and (3.7), does already provide an improved approximation. Without loss of generality all distributions under consideration were normalized, such that their mean values equaled 1. The relative dispersion of a distribution is expressed by its coefficient of variation ν , namely for any Z :

$$\nu(Z) = \frac{\sqrt{\text{Var}(Z)}}{E(Z)} \quad (3.8)$$

He studied the ranges of γ_1 , γ_2 and ν with the above constraints on the mean and variance of $Y(n)$ and found that those parameters were maximized by finding two numbers m_a and m_b and setting:

$$m_1 = m_a,$$

$$m_i = m_b, \quad i = 2, 3, \dots, n. \quad (3.9)$$

Similarly, those parameters were minimized by setting:

$$m_i = m_a, \quad i = 1, 2, \dots, n-1,$$

$$m_n = m_b, \quad (3.10)$$

for the same m_a and m_b . If ν is fixed, then γ_1 and γ_2 are bounded according to

$$2 > \gamma_1^*(n) > \gamma_1 \geq 2\nu > 0,$$

$$6 > \gamma_2^*(n) > \gamma_2 \geq 6\nu > 0. \quad (3.11)$$

Increasing n further increases these maximum values γ_1^* and γ_2^* , whose limits, for $n \rightarrow \infty$ are: $\max \gamma_1 = 2$ and $\max \gamma_2 = 6$. Values of γ_1 and γ_2 as functions of n were computed for the gamma, lognormal, and Weibull distributions, and are presented in Figures 2 and 3 together with the feasible Muth's generalized Erlang distributions. These graphs show that the Erlang distribution can be used to approach the lognormal distribution in the third and fourth moment, if the coefficient of variation is less than 0.6. The Weibull distribution on the other hand cannot be approximated in this fashion.

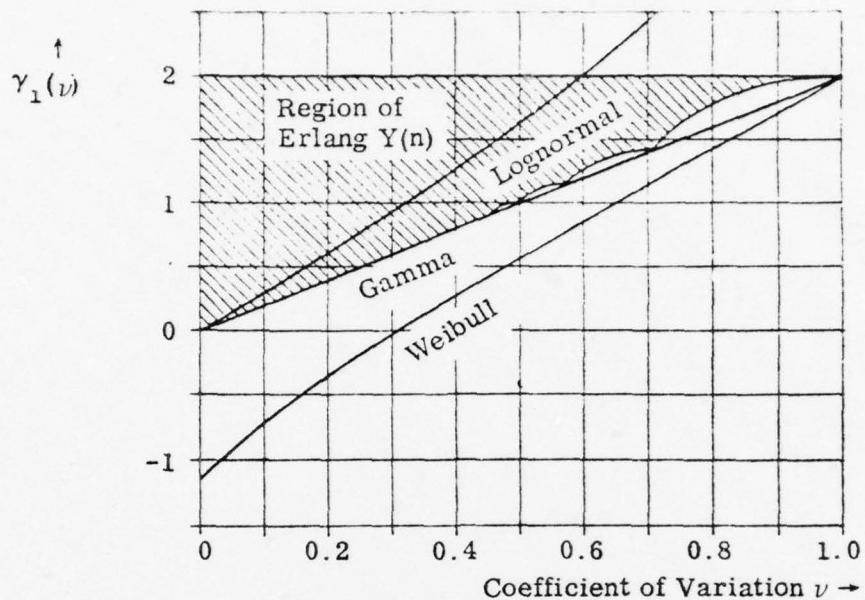


Figure 2: Coefficients of Skewness (from E. J. Muth,
Repairable Systems, Ph.D. Dissertation,
PIB, 1967) .

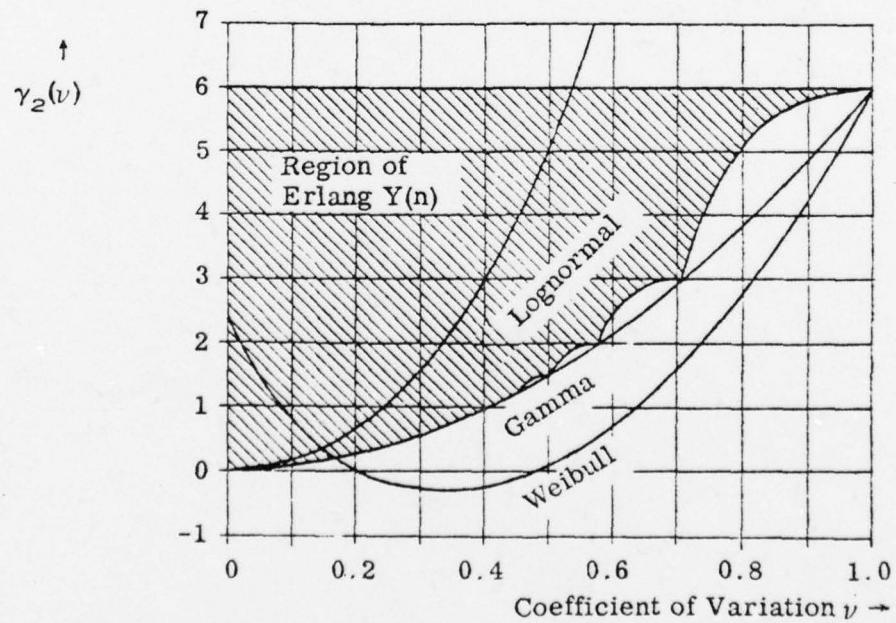


Figure 3: Coefficients of Kurtosis (from E. J. Muth,
Repairable Systems, Ph.D. Dissertation,
PIB, 1967) .

3.2 Sums of Correlated Exponential Variables

The characteristic function of x_n defined in (1.1) as the sum of exponential random variables can be computed as follows. Using (2.15) and the definitions:

$$v = \sum_1^n w_i^2, \quad y = \sum_1^n z_i^2, \quad (3.12)$$

we can write the characteristic functions as:

$$\Phi_{x_n}(s) = \Phi_v(s) \Phi_y(s) = \Phi_v^2(s),$$

due to the independence and identical distributions of w and z . The possibly correlated variables w_i can be represented as linear transformations of independent unit variance normal variables ξ_i :

$$\underline{w} = M \underline{\xi}, \quad E[\underline{\xi} \underline{\xi}'] = I. \quad (3.13)$$

(Here, as throughout this report I is the identity matrix of the appropriate size.) Similarly:

$$\underline{z} = M \underline{\xi}, \quad E[\underline{\xi} \underline{\xi}'] = I. \quad (3.14)$$

It follows that:

$$v = \underline{w}' \underline{w} = \underline{\xi}' M' M \underline{\xi}, \quad (3.15)$$

and:

$$\begin{aligned} \Phi_v(s) &= (\sqrt{2\pi})^{-n} \int \dots \int \exp \left[-\frac{1}{2} \underline{\xi}' \underline{\xi} - s \underline{\xi}' M' M \underline{\xi} \right] d\underline{\xi} \\ &= (\sqrt{2\pi})^{-n} \int \dots \int \exp \left[-\frac{1}{2} \underline{\xi}' R^{-1} \underline{\xi} \right] d\underline{\xi}, \end{aligned} \quad (3.16)$$

where we have defined the matrix:

$$R^{-1} = I + 2s M^T M \quad . \quad (3.17)$$

The integral in (3.16) is of the form of a normal density integrated over all values, except for a scale factor. Thus:

$$\Phi_v(s) = \left(\sqrt{|R^{-1}|} \right)^{-1} \quad , \quad (3.18)$$

and the desired characteristic function for the sum of exponential variables is:

$$\Phi_{x_n}(s) = \left(|I + 2s M^T M| \right)^{-1} = 1/q(s) \quad . \quad (3.19)$$

Equation (3.19) shows that the characteristic function of x_n is the reciprocal of an n^{th} -degree (or less, the rank of Γ is less than or equal to n) polynomial $q(s)$ whose coefficients are determined by the covariance matrix Γ . Properties of that polynomial characterize x_n .

In particular, we have the theorem:

All possible density functions f_{x_n} for $x_n = \sum_1^n r_i$ can be achieved with independent r_i , i.e., with diagonal Γ .

The proof begins by noting that the roots of $q(s)$ are negative reciprocals of the eigenvalues of the symmetric nonnegative definite matrix $2M^T M$ (or equivalently of 2Γ), so they are negative numbers. If the r_i are independent, then M is diagonal:

$$M_I = \text{diag}(\alpha_1, \alpha_2, \dots, \alpha_n) \quad , \quad (3.20)$$

and so is:

$$2M_I M_I = \text{diag}(2\alpha_1^2, 2\alpha_2^2, \dots, 2\alpha_n^2) \quad . \quad (3.21)$$

In this "independent" case :

$$q_I(s) = \frac{n}{\pi} \sum_{i=1}^n (1 + 2\alpha_i^2 s) \quad . \quad (3.22)$$

The proof is completed by comparing the polynomial $q(s)$ in (3.19) for a general M with the $q_I(s)$ in (3.22) for independent r_i . The mean values of the latter ($2\alpha_i^2$) can be chosen to make the roots of $q_I(s)$ match any n (necessarily real) roots of $q(s)$. The resulting polynomials and characteristic functions will be identical because the necessary $\Phi(0) = 1$ property removes any scale factor ambiguity. This completes the proof. ||

In general, the n -independent exponential variables whose sum is indistinguishable from the sum of n -correlated ones will have different mean values from those of the correlated variables.

3.3 Large Number of Summands and Lognormal Approximations

Once this structure has been established for sums of correlated exponential variables, previous results for sums of independent variables (e.g., those of Muth [22]) are directly applicable. He sought, among other results, appropriate mean values for independent exponential variables such that their sum should well approximate a lognormal variable whose density is of the form:

$$f(x) = (x\sigma\sqrt{2\pi})^{-1} e^{-(\ln x - \eta)^2/2\sigma^2}; \quad x \geq 0 \quad , \quad (3.23)$$

with mean and variance:

$$E[x] = e^{\eta + \sigma^2/2} , \quad (3.24)$$

$$\text{Var}[x] = e^{2\eta + \sigma^2} [e^{\sigma^2} - 1] . \quad (3.25)$$

While the theorem above suggests there might be no advantage to allowing correlations among the r_i when the x_n in (1.1) is to approximate x , experience has shown that this new viewpoint can be convenient. Figure 4 shows a lognormal density having $E[x] = 1$ and $\text{Var}[x] = 0.653646$, along with approximations to it.

The approximations in Figure 4 are sums of correlated exponential variables based on underlying normal distributions having covariances of the form of (2.23). Furthermore, the summands are assumed to have equal mean values (all $\sigma_i = \sigma_1$). In this way each x_n is completely characterized by three numbers: n , ρ , σ_1 .

The mean values are chosen to be:

$$E[r_i] = 2\sigma_1^2 \approx \frac{1}{n} , \quad (3.26)$$

so the mean of x_n matches that of x . The correlation parameter ρ in (2.23) is then chosen so that the variances of x_n and x are equal. It is straightforward to compute:

$$\begin{aligned} \text{var}(x_2) &= (1 + \rho^2)/2 , \\ \text{var}(x_3) &= (3 + 4\rho^2 + 2\rho^4)/9 , \\ \text{var}(x_4) &= (2 + 3\rho^2 + 2\rho^3 + \rho^6)/8 . \end{aligned} \quad (3.27)$$

using the underlying normal densities and the mean values determined in (3.26). For the specified $\text{Var}(x) = 0.653646$ it turns out that the correlation factors must be as shown in Table 1.

TABLE I	
Correlation Factors for Normalized f_{x_n} .	
n	ρ
2	0.554339
3	0.750000
4	0.820321

In principle, the densities f_{x_n} can be computed by Laplace transform inversion of (3.19) after the ρ and σ_i values have been specified. An alternative approach is possible once the roots $(-\lambda_i)$ of $q(s)$ have been determined. For example, if all those roots are distinct then:

$$f_{x_n}(t) = \sum_{i=1}^n a_i e^{-\lambda_i t} \quad . \quad (3.28)$$

One condition on the unknown a_i is:

$$\int_0^\infty f_{x_n}(t) dt = 1 = \sum_1^n a_i / \lambda_i \quad . \quad (3.29)$$

Application of the Initial Value Theorem to (3.28) shows that derivatives of the density must satisfy:

$$\begin{aligned} f_{x_n}^{(k)}(0) &= 0 \\ &= \sum_1^n (-\lambda_i)^k a_i; \quad k = 0, 1, \dots, (n-2) . \end{aligned} \quad (3.30)$$

Equations (3.29) and (3.30) provide n-equations for the n-unknowns a_i . This approach has been used to compute the approximating densities, with the following results: (single precision)

$$f_{x_2} = 1.80395 (e^{-1.28672t} - e^{-4.48772t}),$$

$$f_{x_3} = 0.991590 e^{-16.3102t} + 1.67508 e^{-1.26126t} \\ - 2.6667 e^{-6.85714t},$$

$$f_{x_4} = 1.65610 e^{-1.25572t} - 2.69805 e^{-7.79522t} \\ + 1.16646 e^{-21.4009t} - 0.145528 e^{-34.9285t}. \quad (3.31)$$

Figure 4 reveals that as n increases, this particular kind of sum of correlated exponential variables has a distribution which approaches the lognormal shape. Table 2 shows the mean values for the equivalent independent exponential random variables whose sums have the same distributions, respectively. These means are the reciprocals of the exponents in (3.31). There is considerable intuitive appeal to this kind of a limit of summing more and more terms, each contributing equally, on the average, with correlation between two summands decreasing exponentially as their separation increases. However, no precise asymptotic result seems possible. The "heavy-tail" of the lognormal (slower than exponential decay of $f(x)$ for large x) will never be matched by an x_n since, for every n , the tail of $f_{x_n}(x)$ will be dominated by the slowest decay of its (at most n) decaying exponential summands.

Other obstacles to having the distribution of a sum of exponential variables approach the lognormal form are given in Muth's analysis of bounds on the moments of such sums. Those results for independent summands are applicable to correlated ones, due to the theorem presented earlier.

TABLE 2

Means of Independent Exponential Variables Whose Sums Have
Densities f_{x_n} .

n	η_1	η_2	η_3	η_4
2	0.77716	0.2228		
3	0.79286	0.1458	0.0613	
4	0.79636	0.12828	0.04673	0.02863

The foregoing example was based on a covariance matrix of the form of (2.23) in which σ and ρ were chosen so x_n would have prescribed mean and variance. This approach could be generalized to covariance matrices having more adjustable parameters in the hopes of matching more moments of either experimental data or of the lognormal distribution. One example would be the following with parameters σ ,

π_1, π_2 :

$$\gamma_{ij} = (A\pi_1^{|i-j|} + B\pi_2^{|i-j|})\sigma^2 \quad , \quad (3.32)$$

$$|\pi_1 + \pi_2| < 2 \quad ,$$

where:

$$-\frac{(\pi_1 + \pi_2)^2}{4} \leq -\pi_1\pi_2 \leq 1 - |\pi_1 + \pi_2| \quad ,$$

$$A = \frac{\pi_1(1 - \pi_2^2)}{(\pi_1 - \pi_2)(1 + \pi_1\pi_2)} \quad ,$$

$$B = \frac{\pi_2(1 - \pi_1^2)}{(\pi_1 - \pi_2)(1 + \pi_1\pi_2)} \quad .$$

However, Muth's bound on the skewness of a sum of exponentials precludes adjustment of the three parameters here to match the first three moments of every possible distribution.

3.4 Conclusions

A multivariate exponential distribution based on sums of squares of normal variables was examined. We showed that variation in the correlation and means of such variables does not produce any distributions for their sum which could not have been realized by a sum of independent exponential variables. However, sums of such variables, when they have equal means and stationary correlations, tend to have distributions shaped like the lognormal, except in their tail regions.

It should be noted that much of the popularity that the lognormal distribution enjoys for downtime modeling probably results from the easy use of normal probability paper for estimation of lognormal parameters from data. In most cases, other distributions, say f_{x_n} for some small n and suitable σ_1 and ρ , might fit the data equally well. In any event, the peculiar lognormal tail behavior, corresponding to rare events, is not likely to be a necessary property of a distribution being fitted to typical amounts of data.

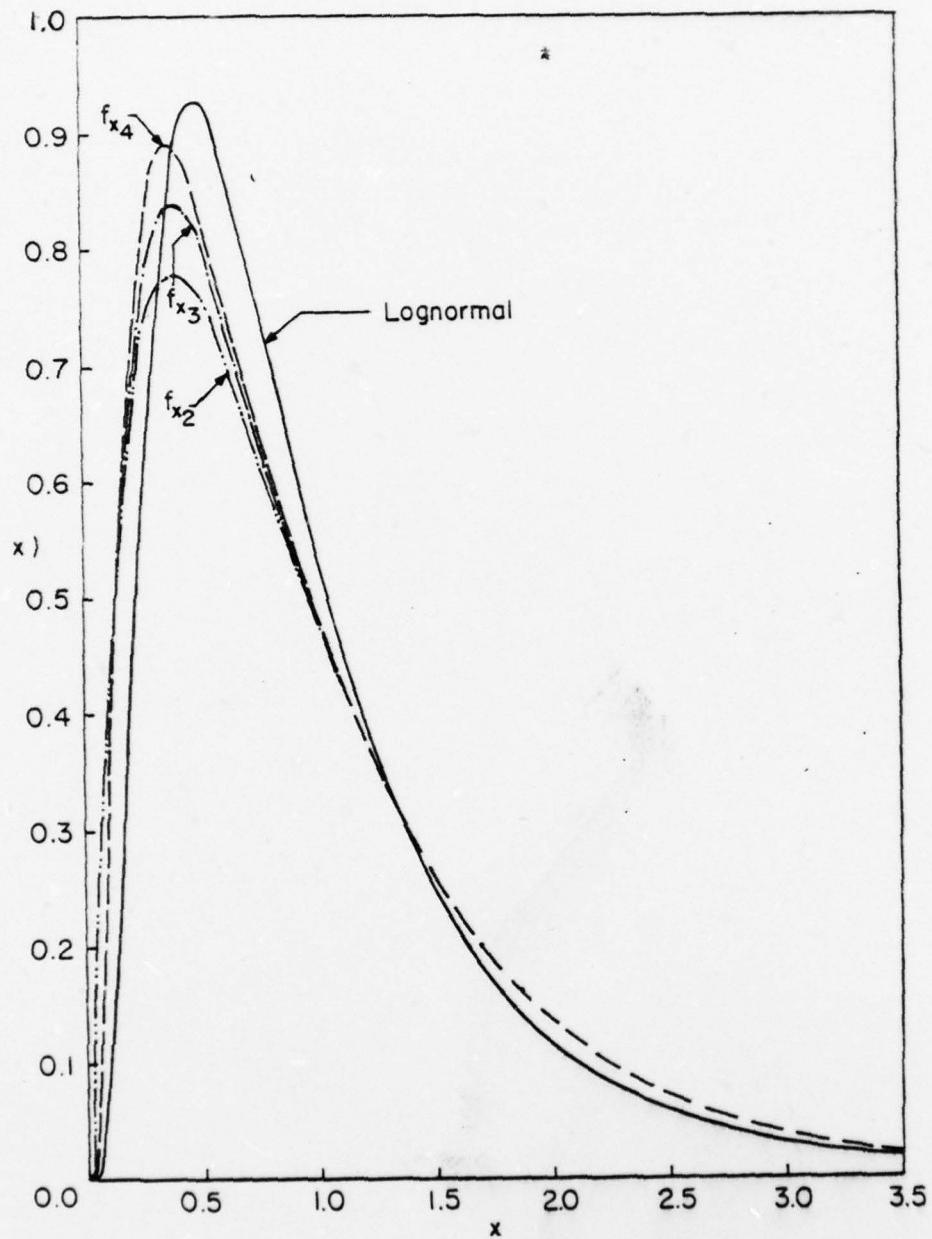


Figure 4: Lognormal Density and Approximations to it

mean	= 1
variance	= 0.653646
coefficient of variation	= 0.808484

CHAPTER IV

OPTIMAL REPLACEMENT OF DETERIORATING PARTS

A single component system is assumed to progress through a finite number of increasingly bad levels of deterioration. The system with level i ($0 \leq i \leq n$) starts in state 0 when new, and is definitely replaced upon reaching the worthless state n . It is assumed that the transition times are directly monitored and the admissible class of strategies allows substitution of a new component only at such transition times. The durations in various deterioration levels are dependent random variables with exponential marginal distributions and a particularly convenient joint distribution. Strategies are chosen to maximize the average rewards per unit time. For some reward functions (with the reward rate depending on the state and the duration in this state) the knowledge of previous state duration provides useful information about the rate of deterioration.

4.1 Review of Literature and Introductory Remarks

Many mathematical studies have been devoted to the optimization of rules for inspecting system quality and for repairing or replacing parts as they are observed to deteriorate. This section reviews some of those previous results and proposes a new model which allows use of a measure of deterioration rate by the controller which replaces parts so as to optimize the average reward per unit time.

Many inspection policy models, where inspections reveal a malfunction occurrence only, have been published; for example, Luss and Kander [19]. Several papers deal with models where deterioration can be observed, and most of them concentrate on replacement policies

which assume that the system's state is always known. Derman [4], [5], Barlow and Proschan [1], and others studied such models assuming that the deterioration process is described by a transition probability matrix of a Markov chain. Kao [16] studied optimal replacement rules when changes of states are semi-Markovian. Rosenfield [26] examined properties of optimal policies for models in which the system's state is observed by inspections only. Kander [15] also examined inspection models. However, he assumed that the operating costs occurring during the system's life do not change with the increasing deterioration.

Luss [20] also examined inspection models, he assumed that the operating costs occurring during the system's life increase with the increasing deterioration. However, he assumed that the holding times in the various states are independently, identically, and exponentially distributed. The policies examined include the scheduling of inspections (when an inspection reveals that the state of the system is better than certain critical state k) and preventive repairs (when an inspection reveals the state of the system being worse than or equal to k). The convenience of a Poisson-type structure for the number of events-per-unit-time made it relatively easy to allow general freedom in the selection of observation times.

The recursive equations of the expected cost to the end of the cycle, incurred subsequent to an inspection at which the system was observed in state i , are:

$$\begin{aligned} L_i &= J_i + P_{ii}(\tau_i)L_i + \sum_{j=i+1}^n P_{ij}(\tau_i)L_j \\ &= [J_i + \sum_{j=i+1}^n P_{ij}(\tau_i)L_i]/P_{ii}^c(\tau_i) \quad (i = 0, 1, \dots, k-1), \end{aligned} \quad (4.1)$$

where :

$$P_{ij}(\tau_i) = \frac{(\lambda \tau_i)^{j-i}}{(j-i)} \exp(-\lambda \tau_i) \quad (i \leq j < n), \quad (4.2)$$

J_i = the expected inspection and occupancy costs to the next event (an inspection or a malfunction) incurred subsequent to an inspection at which the system was observed in state i ,

c = the complementary probability.

An optimum inspection procedure is a specification of the successive inspection times $\{\tau_i^*, 0 \leq i < k\}$ for which the expected cost is a minimum. The total expected cost per unit time $L_0/E(t)$ can be optimized in terms of an auxiliary optimization function (the Breender's method [3], [1]) :

$$D_0 = L_0 - \alpha E(t), \quad (4.3)$$

where $E(t)$ is the expected cycle length. From (4.3) the expected cost is transformed to the following recursive equations:

$$D_i = [J_i - \alpha J'_i + \sum_{j=i+1}^n P_{ij}(\tau_i) D_j] / P_{ii}^c(\tau_i) \quad (i = 0, 1, \dots, k-1), \quad (4.4)$$

where J'_i is the expected time elapsing from an inspection at which the system was observed in state i to the next event (an inspection or a malfunction detection). The minimal value of D_0 , for fixed k and α , is obtained by finding recursively the τ_i^* 's ($i = 0, 1, \dots, k-1$) which minimize D_0 . Since all the τ_j^* 's for $j > i$ are known when D_i is minimized, τ_i^* is obtained by minimizing a nonlinear function of a single variable. By varying α and k , the D_0 is minimized to zero with the optimal α^* and k^* . The minimum total expected cost is obtained and is equal to α^* . The optimal policy $\{k^*; \tau_i^* (i=0, 1, \dots, k-1)\}$

is also obtained.

The work studied here is based on a modification of the model used by Luss. Our model for deterioration is more general, but the admissible strategies used here are more restricted. Here we allow the exponentially distributed durations to have different mean values, and to be positively correlated. The joint distribution assumed for these variables is introduced in Chapter II. Some relationships of dependence among these variables are studied in the next section.

The presence of correlation between interval durations permits the modeling of a rate of deterioration which can be estimated during a particular realization of its past. However, the lack of a Poisson-type of structure for the events-per-unit-time makes it much more difficult here to allow general freedom in the selection of observation times. At present only the simple case of direct and instantaneous observation of deterioration jumps has been considered.

Figure 5 shows a typical time history of deterioration and replacement. The duration in state $(i-1)$, prior to reaching state (i) , is r_{i-1} . The sequence $\{r_i\}$ will be Markov, characterized by a multivariate exponential distribution described in Chapter II and in the next section. Reward functions will be related to the deterioration state and the time spent in each state. The decision rule specifies whether or not to replace, when entering each state i , on the basis of the history of r_{i-1}, r_{i-2}, \dots . The Markov property simplifies the decision rule to a collection of \mathcal{C}_i sets such that we replace on entering state i if and only if $r_{i-1} \in \mathcal{C}_i$.

The objective is to maximize the average reward per unit time:

$$L = \lim_{T \rightarrow \infty} \frac{1}{T} (\text{Total reward in } (0, T)) \quad (4.5)$$

$$= \frac{E[\text{Reward per renewal}]}{E[\text{Duration between renewals}]} = \frac{R}{\theta}. \quad (4.6)$$

(See Ross [27] for equivalence of (4.5) and (4.6).) The mean reward per renewal is defined here as:

$$R = E \left[\sum_{i=0}^{N-1} \int_0^{r_i} c_i(t) dt - p_N \right], \quad (4.7)$$

in which:

N = state at which replacement occurs (possibly random).

p_N = replacement cost if replaced on entering state N (possibly random).

$c_i(t)$ = reward rate when in state i .

Figure 6 shows several reward rate time functions $c(t)$ which have been considered. When one of these $c(t)$ functions is specified for a given problem, the $c_i(t)$ in (4.7) are assigned values $\beta_i c(t)$ with:

$$\beta_0 \geq \beta_1 \geq \beta_2 \geq \dots \geq \beta_{n-1} \geq \beta_n = 0, \quad (4.8)$$

to assure greater reward rates in less deteriorated states.

The mean duration θ in (4.6) is defined as:

$$\theta = E \left[\sum_{i=0}^{N-1} r_i + d_N \right], \quad (4.9)$$

to include a possibly random time d_N for carrying out a replacement at state N .

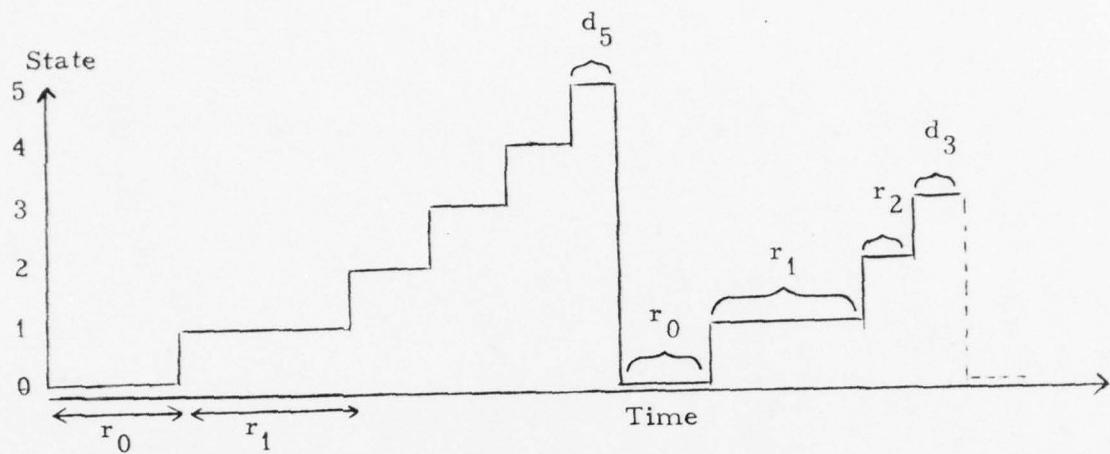


Figure 5:

History of Deterioration and Replacement ($n=5$) .

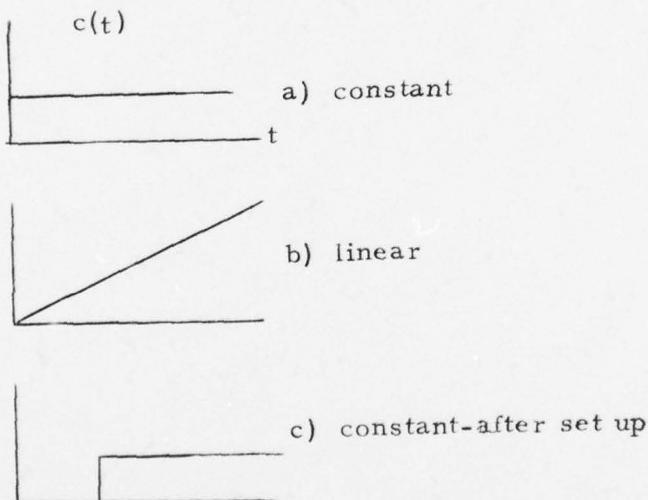


Figure 6:

Reward Rate Time Functions .

While the ultimate objective is to choose C_i to maximize L , it is well known that a related problem of maximizing :

$$L_0(\alpha) = R - \alpha D \quad , \quad (4.10)$$

is simpler [1]. Indeed, the C_i which maximize L will be identical to those which maximize $L_0(\alpha)$ for the α^* such that:

$$L_0^0(\alpha^*) = 0, \text{ where } L_0^0(\alpha) \triangleq \max_{\{C_i\}} L_0(\alpha) . \quad (4.11)$$

Section 4.3 considers the constant reward rate case in which it is found that deterioration rate information is not useful (e.g., the optimal policy is independent of the amount of correlation between successive state durations).

Sections 4.4 and 4.5 consider other reward rate structures for which the optimal policies do make use of estimates of the deterioration rates as well as of observations of the deterioration level.

The next section develops interesting dependence properties of the $\{r_i\}$ sequence which will be needed in the optimization studies.

4.2 Relationships of Dependence Among Multivariate Exponential Variables

For easier and more convenient notations, we make the following changes in Equations (2.10) and (2.25) with means :

$$E[r_i] = \eta_i, \quad (4.12)$$

and correlation coefficients :

$$\rho_{r_i, r_j} = \rho^{|i-j|}. \quad (4.13)$$

Therefore we have

$$f(r_{i-1}, r_i) = \frac{1}{(1-\rho)\eta_{i-1}\eta_i} \left[e^{-\frac{r_{i-1}}{(1-\rho)\eta_{i-1}} - \frac{r_i}{(1-\rho)\eta_i}} \right] \cdot I_0 \left[\frac{2}{1-\rho} \frac{\sqrt{\rho}}{\sqrt{\eta_i \eta_{i-1}}} \sqrt{r_{i-1} r_i} \right], \quad (4.14)$$

and:

$$f(r_0, r_1, r_2, \dots, r_{n-1}) = \left[(1-\rho)^{n-1} \prod_{i=0}^{n-1} \eta_i \right]^{-1} \cdot \prod_{i=0}^{n-2} I_0 \left[\frac{2}{1-\rho} \frac{\sqrt{\rho}}{\sqrt{\eta_i \eta_{i+1}}} \sqrt{r_i r_{i+1}} \right] \\ \cdot \exp \left[-\frac{1}{(1-\rho)} \left(\frac{r_0}{\eta_0} + \frac{r_{n-1}}{\eta_{n-1}} + \sum_{i=1}^{n-2} \frac{r_i(1+\rho)}{\eta_i} \right) \right]; \quad n > 2. \quad (4.15)$$

From our stationary correlation structure, we have a Markov sequence. The conditional density function is easily determined from Equation (4.15):

$$f(r_i | r_{i-1}, r_{i-2}, \dots, r_0) = f(r_i | r_{i-1}). \quad (4.16)$$

The Markov property will simplify replacement schedule optimization, a procedure which will also make use of other properties of the conditional densities:

$$f(r_i | r_{i-1}) = \left[\eta_i(1-\rho) \right]^{-1} \exp \left[-\frac{1}{(1-\rho)} \left(\frac{r_i}{\eta_i} + \frac{\rho r_{i-1}}{\eta_{i-1}} \right) \right] \cdot I_0 \left(\frac{2}{1-\rho} \sqrt{\frac{\rho r_i r_{i-1}}{\eta_i \eta_{i-1}}} \right). \quad (4.17)$$

For example, it can be shown [23] that:

$$E[r_i | r_{i-1}] = \eta_i + (r_{i-1} - \eta_{i-1})\rho \quad \eta_i / \eta_{i-1}, \quad (4.18)$$

$$\text{Var}[r_i | r_{i-1}] = \eta_i^2 [(\mathbf{1}-\rho)^2 + 2\rho(1-\rho) r_{i-1} / \eta_{i-1}] . \quad (4.19)$$

These conditional moments show, e.g., that the conditional mean of r_i exceeds its mean in proportion to the amount by which r_{i-1} exceeds its mean.

Before deriving some dependent relations among r_i and r_{i-1} , we review the relationships among some notions of multivariate dependence [2].

Definitions (4.1): Given random variables S and T, we say the following: (Definitions 5.4.1 of [2])

(a) T is stochastically increasing in S if :

$$P[T > t | S = s] , \quad (4.20)$$

is increasing in s for all t. We write $SI(T/S)$.

(b) Let S and T have joint probability density $f(s, t)$. Then $f(s, t)$ is totally positive of order 2 if :

$$\begin{vmatrix} f(s_1, t_1) & f(s_1, t_2) \\ f(s_2, t_1) & f(s_2, t_2) \end{vmatrix} \geq 0 , \quad (4.21)$$

for all $s_1 < s_2$, $t_1 < t_2$ in the domain of S and T. We write $f(s, t)$ is TP_2 or, alternately, $TP_2(S, T)$.

Theorem (4.1): $TP_2(S, T) \Rightarrow SI(T/S)$ (Theorem 5.4.2 of [2])

Proof: From (4.21), we obtain :

$$\left| \begin{array}{cc} \int_t^\infty f(s_1, \tau) d\tau & \int_t^\infty f(s_2, \tau) d\tau \\ \int_0^t f(s_1, \tau) d\tau & \int_0^t f(s_2, \tau) d\tau \end{array} \right| \leq 0 \quad , \quad (4.22)$$

for $s_1 < s_2$. Adding the top row to the bottom row and converting to ratios, we obtain the inequality for conditional probabilities:

$$P[T > t | S = s_1] \leq P[T > t | S = s_2]. \quad (4.23)$$

Thus $SI(T/S)$ holds. ||

The following theorems will help us understand intuitively the information about the rate of deterioration contained in the past observation.

Another statement in the same spirit of Equation (4.18) (i.e., that a large r_{i-1} implies r_i will tend to be large) is that r_i is stochastically increasing in r_{i-1} . In order to prove $SI(r_i | r_{i-1})$, it suffices to show that $f(r_i, r_{i-1})$ satisfies the TP_2 condition.

Theorem (4.2): r_i is stochastically increasing in r_{i-1} .

Proof: By substitution of Equation (4.14) into the determinant (4.21) and removing the non-negative factors it remains to be shown that :

$$Det. = \left| \begin{array}{cc} I_0(\xi \sqrt{s_1 t_1}) & I_0(\xi \sqrt{s_1 t_2}) \\ I_0(\xi \sqrt{s_2 t_1}) & I_0(\xi \sqrt{s_2 t_2}) \end{array} \right| \geq 0 , \quad (4.24)$$

where $\xi = 2 \sqrt{\rho} / [(1-\rho) \sqrt{n_i n_{i-1}}]$. With the power series expansion of Bessel functions, each Bessel function can be expanded according to the power series :

$$I_0(z) = \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2k}}{(k!)^2}.$$

Det. can be written as a difference of products of infinite summations:

$$\begin{aligned} \text{Det.} &= \left| \begin{array}{cc} \sum_{i=0}^{\infty} \frac{\left(\frac{\epsilon \sqrt{s_1 t_1}}{2}\right)^{2i}}{(i!)^2} & \sum_{j=0}^{\infty} \frac{\left(\frac{\epsilon \sqrt{s_1 t_2}}{2}\right)^{2j}}{(j!)^2} \\ \sum_{i=0}^{\infty} \frac{\left(\frac{\epsilon \sqrt{s_2 t_1}}{2}\right)^{2i}}{(i!)^2} & \sum_{j=0}^{\infty} \frac{\left(\frac{\epsilon \sqrt{s_2 t_2}}{2}\right)^{2j}}{(j!)^2} \end{array} \right| \\ &= \sum_{i=0}^{\infty} \frac{\left(\frac{\epsilon \sqrt{s_1 t_1}}{2}\right)^{2i}}{(i!)^2} \cdot \sum_{j=0}^{\infty} \frac{\left(\frac{\epsilon \sqrt{s_2 t_2}}{2}\right)^{2j}}{(j!)^2} - \sum_{i=0}^{\infty} \frac{\left(\frac{\epsilon \sqrt{s_2 t_1}}{2}\right)^{2i}}{(i!)^2} \cdot \sum_{j=0}^{\infty} \frac{\left(\frac{\epsilon \sqrt{s_1 t_2}}{2}\right)^{2j}}{(j!)^2}. \end{aligned}$$

For each pair of non-negative integers (i, j) , the expanded version of Det. will contain four summands with factors $[i! j!]^{-2}$ (if $i \neq j$), and two such terms if $i = j$. If $i = j + m$, $m > 0$, then the four terms can be written as:

$$\left[\frac{\rho}{\eta_1 \eta_2 (1-\rho)^2} \right]^{2(i+j)} [i! j!]^{-2} (s_1 s_2 t_1 t_2)^j \{ (s_1 t_1)^m + (s_2 t_2)^m - (s_1 t_2)^m - (s_2 t_1)^m \}. \quad (4.25)$$

The sign of (4.25) will be determined by the factor in brackets $\{ \cdot \}$. Noting that for $0 \leq s_1 \leq s_2$, $0 \leq t_1 \leq t_2$, and m a non-negative integer:

$$\sigma_1 = s_1^m \leq \sigma_2 = s_2^m,$$

$$\tau_1 = t_1^m \leq \tau_2 = t_2^m.$$

Since:

$$\sigma_1 \tau_1 + \sigma_2 \tau_2 \geq \sigma_1 \tau_2 + \sigma_2 \tau_1$$

when:

$$\sigma_1 \leq \sigma_2 ,$$

$$\tau_1 \leq \tau_2 ,$$

it is clear that (4.25) is non-negative. Finally, Det. is seen to be non-negative because the infinite summation can be grouped into terms like (4.25), or into the corresponding $i = j$ terms which vanish. This completes the proof that $f(r_i, r_{i-1})$ is TP_2 , which implies that r_i is stochastically increasing in r_{i-1} . ||

The $SI(r_i | r_{i-1})$ property will be used in the optimization derivations later, in conjunction with another theorem:

Theorem (4.3): If $SI(x|y)$ and $h(x)$ is an increasing function of x , then $E[h(x)|y]$ is increasing in y . (Proposition 3.1 on p. 22 of [17].)

Proof: For $y_1 > y_2$:

$$E[h(x)|y_1] - E[h(x)|y_2] = \int_{-\infty}^{\infty} h(x)[f(x|y_1) - f(x|y_2)] dx.$$

But by assumption:

$$\int_t^{\infty} [f(x|y_1) - f(x|y_2)] dx = P[x > t|y_1] - P[x > t|y_2] \geq 0$$

for all t ,

also:

$$\int_{-\infty}^{\infty} [f(x|y_1) - f(x|y_2)] dx = 0.$$

These expressions allow us to invoke Lemma 1 and its corollary on p. 120 of [2], viz.:

If $W(x)$ is a Lebesgue-Stieltjes measure, not necessarily positive, for which:

$$\int_t^\infty dW(x) \geq 0 \quad \text{for all } t,$$

$$\int_{-\infty}^\infty dW(x) = 0,$$

and $h(x)$ is increasing, then $\int_{-\infty}^\infty h(x)dW(x) \geq 0$. This completes the proof. ||

Another interesting property is that the random variables r_0, r_1, \dots, r_{n-1} are conditionally increasing in sequence. This is equivalent to saying that r_i is stochastically increasing in r_{i-1}, \dots, r_1, r_0 for $i = 1, \dots, n-1$, i.e.

$$P[r_i > r_i^* | r_{i-1}, r_{i-2}, \dots, r_1, r_0] ,$$

is increasing in $r_{i-1}, r_{i-2}, \dots, r_1, r_0$.

Theorem (4.4): If the random variables r_0, r_1, \dots, r_{n-1} obey the joint density function in Equation (4.15), then r_0, r_1, \dots, r_{n-1} are stochastically increasing in sequence.

Proof: We invoke the Theorem (5.4.14) of [2], viz.:

Let r_0, \dots, r_{n-1} have joint density $f_n(r_0, \dots, r_{n-1})$ which is TP_2 in each pair of arguments for fixed values of the remaining arguments. Then r_0, \dots, r_{n-1} are conditionally increasing in sequence.

In order to prove that $f(r_i, r_{i-1} | r_{i-2} = c_{i-2}, \dots, r_0 = c_0)$ is a TP_2 (r_i, r_{i-1}), it suffices to show that $f(r_i, r_{i-1} | r_{i-2} = c_{i-2})$ is a TP_2 (r_i, r_{i-1}) because of the Markov property of the $\{r_i\}$ sequence. We have:

$$f(r_i, r_{i-1} | r_{i-2} = c_{i-2}) = f(r_i | r_{i-1}) f(r_{i-1} | r_{i-2} = c_{i-2}) \\ = f(r_i, r_{i-1}) \cdot \frac{f(r_{i-1} | r_{i-2} = c_{i-2})}{f(r_{i-1})},$$

and:

$$\begin{vmatrix} f(s_1, t_1 | c_{i-2}) & f(s_1, t_2 | c_{i-2}) \\ f(s_2, t_1 | c_{i-2}) & f(s_2, t_2 | c_{i-2}) \end{vmatrix} \\ = \frac{f(t_1 | c_{i-2})}{f(t_1)} \cdot \frac{f(t_2 | c_{i-2})}{f(t_2)} \cdot \begin{vmatrix} f(s_1, t_1) & f(s_1, t_2) \\ f(s_2, t_1) & f(s_2, t_2) \end{vmatrix},$$

which is always nonnegative from Theorem (4.2). Therefore we have $TP_2(r_i, r_{i-1})$. For those random variables which are not next to each other, such as r_i and r_j for $j \neq i-1$, they are conditionally independent by the Markov property of the $\{r_i\}$ sequence. This implies $TP_2(r_i, r_j)$ and completes the proof. ||

The stochastically increasing property is not sufficient for our analytical purposes. To show that the conditional density function of r_i given r_{i-1} is a convexity-preserving transformation [17], a property which we shall need in this chapter, one method is to show that $TP_3(r_i, r_{i-1})$ is true. However, a direct proof which requires a 3 by 3 matrix expansion of Bessel functions is quite complicated. Therefore other methods are sought after. We summarize several important facts and theorems in the theory of total positivity [17] that will be extensively used later.

Definition (4.2): A real function $K(x, y)$ of two variables varying over linearly ordered sets X and Y , respectively, is said to be totally positive of order r (abbr. TP_r) if for all:

$$x_1 < x_2 < \dots < x_m, y_1 < y_2 < \dots < y_m, x_i \in X, y_j \in Y, 1 \leq m \leq r, \quad (4.26)$$

we have the inequalities:

$$K\begin{pmatrix} x_1, x_2, \dots, x_m \\ y_1, y_2, \dots, y_m \end{pmatrix} = \begin{vmatrix} K(x_1, y_1) & K(x_1, y_2) & \dots & K(x_1, y_m) \\ K(x_2, y_1) & K(x_2, y_2) & \dots & K(x_2, y_m) \\ \vdots & \vdots & \ddots & \vdots \\ K(x_m, y_1) & K(x_m, y_2) & \dots & K(x_m, y_m) \end{vmatrix} \geq 0.$$

(4.27)

If strict inequality holds, then we say that K is strictly totally positive of order r (abbr. STP_r).

Remark: From the condition (4.26), we know that if $K(x, y)$ is TP_r then it is also TP_m for $1 \leq m \leq r$.

For illustration, we begin by citing one basic example [17]. The function:

$$K(x, y) = e^{\phi(x)} \psi(y), \quad (4.28)$$

is $TP_{\infty}(STP_{\infty})$ on $X \times Y$, provided $\phi(x)$ and $\psi(y)$ are increasing functions (strictly) on the sets of X and Y , respectively, of the real line.

Lemma (4.1): If $L(\xi, \eta)$ is TP_r and $M(\eta, \zeta)$ is TP_s , then:

$$K(\xi, \zeta) = \int_Y L(\xi, \eta) M(\eta, \zeta) d\sigma(\eta) \quad \xi \in X, \zeta \in Z,$$

is $TP_{\min(r, s)}$. (Lemma 3.1.1. (a) of [17].)

Theorem (4.5): If $K(x, y)$ is TP_r , and $\phi(x)$ and $\psi(y)$ are nonzero positive functions for $x \in X$ and $y \in Y$, respectively, and if $L(x, y) = K(x, y) \phi(x)\psi(y)$, then $L(x, y)$ is also TP_r . (Theorem 3.1.1.(a) of [17].)

A direct specification of Lemma (4.1) leads to the conclusion that:

$$K(x, y) = \int_{-\infty}^{\infty} e^{u(x)\alpha(s)} e^{v(y)\beta(s)} d\sigma(s), \quad x \in X, \quad y \in Y, \quad (4.29)$$

is TP_{∞} provided u, v, α , and β are monotonic increasing functions, σ is a sigma-finite positive measure, and (4.29) exists absolutely. (Eq. (3.1.9) of [17].)

Theorem (4.6): $K(x, y) = I_{\alpha}(xy)$ is TP_{∞} for $0 \leq x, y < \infty$, where I_{α} denotes the modified Bessel function of order α . (Example on p. 101 of [17].)

Proof: The modified Bessel function can be expanded as:

$$\begin{aligned} I_{\alpha}(xy) &= \left(\frac{xy}{2}\right)^{\alpha} \cdot \sum_{n=0}^{\infty} \frac{(xy/2)^{2n}}{n! \cdot \Gamma(\alpha+n+1)} \\ &= \left(\frac{x}{\sqrt{2}}\right)^{\alpha} \cdot \left(\frac{y}{\sqrt{2}}\right)^{\alpha} \cdot \sum_{n=0}^{\infty} a_n (x^2)^n (y^2)^n \\ &= \left(\frac{x}{\sqrt{2}}\right)^{\alpha} \cdot \left(\frac{y}{\sqrt{2}}\right)^{\alpha} \int_0^{\infty} \exp(t \log x^2) \exp(t \log y^2) d\sigma(t), \end{aligned} \quad (4.30)$$

where $a_n = [2^{2n} \cdot n! \cdot \Gamma(\alpha+n+1)]^{-1}$ and $\sigma(t)$ is a sigma-finite discrete measure concentrating mass a_n at $t = n$. The integral has the same form as that of Equation (4.29). With $t, \log x^2$, and $\log y^2$ being monotonic increasing functions, we have shown that $I_{\alpha}(xy)$ is TP_{∞} from

Lemma (4.1) and Theorem (4.5). ||

Theorem (4.7): The joint density function (4.14) of r_i and r_{i-1} ,
 $i = 1, 2, \dots, n-1$, is TP_∞ for $0 \leq r_i, r_{i-1} < \infty$.

Proof: We have :

$$f(r_{i-1}, r_i) = \frac{1}{(1-\rho)\eta_{i-1}\eta_i} \left[e^{-\frac{r_{i-1}}{(1-\rho)\eta_{i-1}} - \frac{r_i}{(1-\rho)\eta_i}} \right] \\ \cdot I_0 \left[\frac{2}{1-\rho} \frac{\sqrt{\rho}}{\sqrt{\eta_i \eta_{i-1}}} \sqrt{r_{i-1} r_i} \right],$$

and $I_0[\xi \sqrt{r_{i-1} r_i}]$ where $\xi = \frac{2}{1-\rho} \frac{\sqrt{\rho}}{\sqrt{\eta_i \eta_{i-1}}}$, is TP_∞ from Theorem (4.6).

Therefore, with :

$$\phi(r_{i-1}) = \frac{1}{\sqrt{1-\rho} \eta_{i-1}} e^{-\frac{r_{i-1}}{(1-\rho)\eta_{i-1}}},$$

and:

$$\psi(r_i) = \frac{1}{\sqrt{1-\rho} \eta_i} e^{-\frac{r_i}{(1-\rho)\eta_i}},$$

we have :

$$f(r_{i-1}, r_i) = \phi(r_{i-1}) \psi(r_i) I_0(\xi \sqrt{r_{i-1} r_i}) .$$

Moreover, we note that $\phi(r_{i-1})$ and $\psi(r_i)$ are nonzero positive functions
for $r_{i-1} \in [0, \infty)$ and $r_i \in [0, \infty)$, respectively. Hence $f(r_{i-1}, r_i)$ is a
 TP_∞ from Theorem (4.5). ||

The conditional density function $f(r_i | r_{i-1})$ has the same structure as the joint density function $f(r_{i-1}, r_i)$. It is simple to prove the following theorem:

Theorem (4.8): The conditional density function (4.17) of r_i on r_{i-1} , $i = 1, 2, \dots, n-1$ is TP_∞ for $0 \leq r_i, r_{i-1} < \infty$.

Proof: We have :

$$f(r_i | r_{i-1}) = \frac{1}{(1-\rho)\eta_i} \left[e^{-\frac{r_i}{(1-\rho)\eta_i} - \frac{\rho r_{i-1}}{(1-\rho)\eta_{i-1}}} \right] \cdot I_0(\xi \sqrt{r_i r_{i-1}}),$$

where $\xi = \frac{2}{1-\rho} \frac{\sqrt{\rho}}{\sqrt{\eta_i \eta_{i-1}}}$ and $I_0(\xi \sqrt{r_i r_{i-1}})$ is TP_∞ from Theorem (4.6).

Therefore, with :

$$\phi_c(r_{i-1}) = e^{-\frac{\rho r_{i-1}}{(1-\rho)\eta_{i-1}}} \quad \text{and} \quad \psi_c(r_i) = \frac{1}{(1-\rho)\eta_i} e^{-\frac{r_i}{(1-\rho)\eta_i}},$$

we have :

$$f(r_i | r_{i-1}) = \phi_c(r_{i-1}) \psi_c(r_i) I_0(\xi \sqrt{r_i r_{i-1}}).$$

Since $\phi_c(r_{i-1})$ and $\psi_c(r_i)$ are nonzero positive functions for $r_{i-1} \in [0, \infty)$ and $r_i \in [0, \infty)$, respectively, we know that $f(r_i | r_{i-1})$ is the TP_∞ from Theorem (4.5). ||

Remark: That $f(r_i | r_{i-1})$ is TP_∞ can also be justified by simply observing that it comes from the probability transition function of a diffusion process. See Karlin and McGregor [30, 31].

The following theorem concerns convexity-preserving transformations which we shall use in analyzing our problem.

4.3 Constant Reward Rate

The constant reward rate case with $c_i(t) = \beta_i$ is particularly simple to analyze. We will see that as long as $E[r_i | r_{i-1}, r_{i-2}, \dots] \geq 0$ for all i , even if the r_i are not exponentially distributed, the optimal rule will be to replace upon entering some critical state k^* , independent of the observed durations r_i . For simplicity of exposition we assume that all $p_i = p$ and all $d_i = d$.

Based on the problem statement in Section 4.1 the optimal decision on entering state j must maximize the mean future reward until the next renewal, $\mathcal{L}_j(\alpha)$, for a suitable α . Here:

$$\mathcal{L}_j(\alpha) = E \left[\sum_{i=j}^{N-1} \beta_i r_i \mid r_{j-1} \right] - \alpha E \left[\sum_{i=j}^{N-1} r_i \mid r_{j-1} \right] - p - \alpha d. \quad (4.31)$$

The optimal decisions for each state will be found in terms of α , and then the proper α^* (for producing decisions which maximize \mathcal{L}) is the one for which the maximum:

$$\max \mathcal{L}_0(\alpha^*) \equiv \mathcal{L}_0^0(\alpha^*) = 0. \quad (4.32)$$

Optimization by dynamic programming begins with considering the decision at the last step, i.e., on entering state $(n-1)$. There are two choices, to replace (R) or not to replace (\bar{R}), with corresponding values:

$$\mathcal{L}_{n-1}(\alpha; R) = -p - \alpha d, \quad (4.33)$$

and:

$$\begin{aligned} \mathcal{L}_{n-1}(\alpha; \bar{R}) &= E[\beta_{n-1} r_{n-1} \mid r_{n-2}] - \alpha E[r_{n-1} \mid r_{n-2}] - p - \alpha d \\ &= E[(\beta_{n-1} - \alpha)r_{n-1} \mid r_{n-2}] - p - \alpha d. \end{aligned} \quad (4.34)$$

Clearly, the best decision is not to replace if and only if:

$$\begin{aligned}\Delta_{n-1}(\alpha; r_{n-2}) &\triangleq \mathcal{L}_{n-1}(\alpha; \bar{R}) - \mathcal{L}_{n-1}(\alpha; R) \\ &= (\beta_{n-1} - \alpha) E[r_{n-1} | r_{n-2}] \geq 0.\end{aligned}\quad (4.35)$$

The sign of (4.35) will be the sign of $(\beta_{n-1} - \alpha)$, due to the non-negativity of all interval durations. Thus the best decision depends on α and the reward parameters β_{n-1} but not on the previously observed duration. Two cases will be considered separately.

If $\beta_{n-1} > \alpha$ then the best decision at state $(n-1)$ is not to replace.

We will now explain why, under this condition, it is best not to replace at any state less than n . Consider the situation on entering $(n-2)$. We have already shown that it is best not to replace on entering $(n-1)$. Thus the choice will be based on a Δ_{n-2} of the form:

$$\Delta_{n-2}(\alpha; r_{n-3}) = E[(\beta_{n-2} - \alpha)r_{n-2} + (\beta_{n-1} - \alpha)r_{n-1} | r_{n-3}]. \quad (4.36)$$

Here we have :

$$(\beta_{n-2} - \alpha) > (\beta_{n-1} - \alpha) > 0, \quad (4.37)$$

by assumption, and :

$$E[r_{n-1} | r_{n-3}] \quad \text{and} \quad E[r_{n-2} | r_{n-3}] \geq 0, \quad (4.38)$$

because all $r_i \geq 0$ with probability one. Thus $\Delta_{n-2}(\alpha; r_{n-3}) > 0$ for all $r_{n-3} > 0$, and it is best not to replace here, either. This argument can be repeated for states $(n-3), (n-4), \dots, 1, 0$.

The other case to consider is $\beta_{n-1} < \alpha$ which requires replacement on entering state $(n-1)$, if the system ever reaches that state.

When we consider the decision on entering $(n-2)$, the Δ_{n-2} is:

$$\Delta_{n-2}(\alpha; r_{n-3}) = E[(\beta_{n-2}-\alpha)r_{n-2}|r_{n-3}], \quad (4.39)$$

which has the sign of $(\beta_{n-2}-\alpha)$. If $(\beta_{n-2}-\alpha) < 0$, then replacement is optimal on entering $(n-2)$ and $(n-3)$ is considered next. This iteration may eventually reach a state $(k-1)$ where $(\beta_{k-1}-\alpha) > 0$ it is best not to replace. Arguments similar to those for the $\beta_{n-1}-\alpha > 0$ case show that non-replacement is optimal at all states preceding the one which first arises in this backward iteration as a non-replacement state.

In summary, in the constant reward rate case $\mathcal{L}_0^0(\alpha)$ is maximized by a decision rule which says replace on entering some state $k \leq n$ which depends on the reward parameters $\{\beta_i\}$ and the α :

$$k = \min\{i: (\alpha - \beta_i) > 0\}. \quad (4.40)$$

Finally, we must choose α^* so that $\mathcal{L}_0^0(\alpha^*) = 0$, where:

$$\mathcal{L}_0^0(\alpha) = -p - \alpha d + \sum_0^{k-1} (\beta_i - \alpha) E[r_i] \quad (4.41)$$

Figure 7 shows a typical plot of $\mathcal{L}_0^0(\alpha)$ as a continuous, piecewise linear curve whose zero crossing ($\mathcal{L}_0^0(\alpha^*) = 0$) defines α^* and the optimal replacement state k^* for maximizing L .

Example: Figure 7 shows that the optimal average reward per unit time is $2\frac{5}{7}$ when $k^* = 3$, where $\beta_0 = 5$, $\beta_1 = 4$, $\beta_2 = 3$, $\beta_3 = 2$, $\beta_4 = 1$, $\beta_5 = 0$, $p = 5$, $d = 1$, $\eta_i = 2$ ($i = 0, 1, 2, 3, 4$) and $n = 5$. From Equation (4.41), we have the following values for $\mathcal{L}_0^0(k(\alpha), \alpha)$, the optimal k is a function of α , which remains constant when α varies over each interval $\beta_{i+1} < \alpha < \beta_i$:

$$\mathcal{L}_0(5, 0) = 25,$$

$$\mathcal{L}_0(4, 1) = 14,$$

$$\mathcal{L}_0(3, 2) = 5,$$

$$\mathcal{L}_0(2, 3) = -2,$$

$$\mathcal{L}_0(1, 4) = -7,$$

and the optimal $\mathcal{L}_0^0(k^* = 3, \alpha^* = 2\frac{5}{7}) = 0$.

4.4 Linear Reward Rate

The linear reward rate case with $c_i(t) = 2\beta_i t$ has some very interesting results. By utilizing the stochastically increasing property of $f(r_i | r_{i-1})$, the optimal rule will be seen to depend on the observed durations $\{r_i\}$. For simplicity we also assume that all $p_i = p$ and all $d_i = d$.

Based on the problem statement in Section 4.1, the optimal decision on entering state j must maximize the mean future reward until the next renewal, i.e., $\mathcal{L}_j(\alpha)$. For a suitable α , we have:

$$\mathcal{L}_j(\alpha) = E \left\{ \sum_{i=j}^{N-1} \beta_i r_i^2 \mid r_{j-1} \right\} - \alpha E \left\{ \sum_{i=j}^{N-1} r_i \mid r_{j-1} \right\} - p - \alpha d. \quad (4.42)$$

The optimal decisions for each state will be found in terms of α , and then the proper α^* (for producing decisions which maximize L) is the case for which the maximum :

$$\mathcal{L}_0^0(\alpha^*) = -p - \alpha^* d + E \left[\sum_{i=0}^{N-1} \beta_i r_i^2 - \alpha^* \sum_{i=0}^{N-1} r_i \right] = 0. \quad (4.43)$$

Optimization by dynamic programming begins with considering the

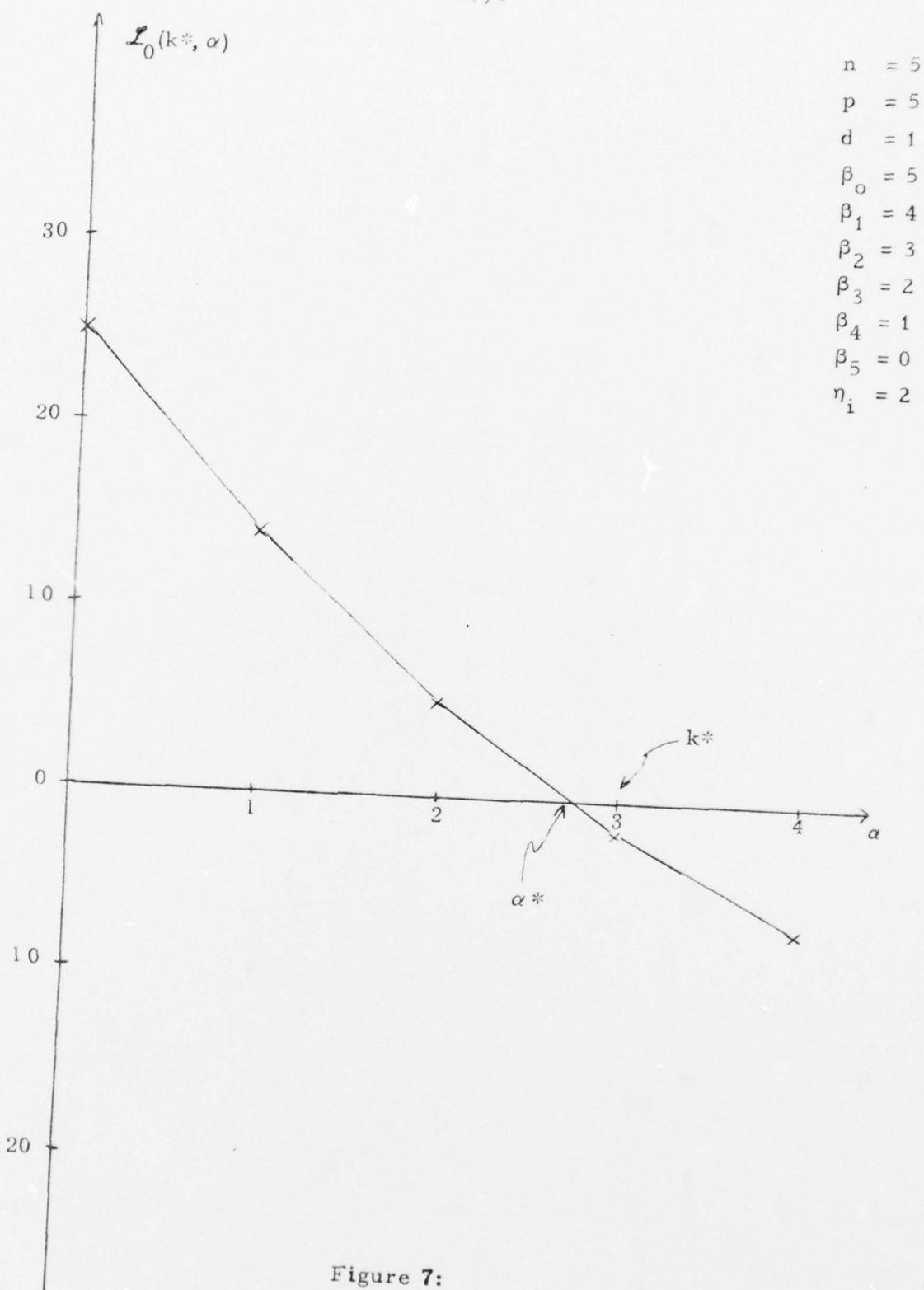


Figure 7:
Constant Reward Rate Case and the
Optimal Average Reward Rate = α^* .

decision at the last step. Since state n represents a failed component, we definitely replace it when it enters state n . Next, we consider the decision to be made on entering state $n-1$. There are two choices: to replace (R) or not to replace (\bar{R}), with corresponding values:

$$\mathcal{L}_{n-1}(\alpha; R) = -p - \alpha d , \quad (4.44)$$

$$\mathcal{L}_{n-1}(\alpha; \bar{R}) = E[\beta_{n-1} r_{n-1}^2 - \alpha r_{n-1} | r_{n-2}] - p - \alpha d , \quad (4.45)$$

for $\mathcal{L}_{n-1}(\alpha)$. Clearly, the best decision is not to replace if and only if:

$$\begin{aligned} \Delta_{n-1}(\alpha) &\stackrel{\Delta}{=} \mathcal{L}_{n-1}(\alpha; \bar{R}) - \mathcal{L}_{n-1}(\alpha, R) \\ &= E[\beta_{n-1} r_{n-1}^2 - \alpha r_{n-1} | r_{n-2}] \\ &\stackrel{\Delta}{=} \bar{\Delta}_{n-1}(r_{n-2}, \beta_{n-1}, \alpha) \geq 0 . \end{aligned} \quad (4.46)$$

Substituting in the conditional moments (see (4.18) and (4.19)), we obtain:

$$\begin{aligned} \bar{\Delta}_{n-1}(r_{n-2}, \beta_{n-1}, \alpha) &= \beta_{n-1} \left(\frac{\eta_{n-1}}{\eta_{n-2}} \right)^2 \rho^2 r_{n-2}^2 \\ &+ \left\{ \beta_{n-1} \frac{(2\eta_{n-1})^2}{\eta_{n-2}} \rho(1-\rho) - \alpha \rho \frac{\eta_{n-1}}{\eta_{n-2}} \right\} r_{n-2} \\ &+ \left\{ \beta_{n-1} - 2\eta_{n-1}^2(1-\rho)^2 \right\} - \alpha \eta_{n-1}(1-\rho) . \end{aligned} \quad (4.47)$$

This quadratic function of r_{n-2} can have the possible convex shapes shown in Figure 8.

Examination of (4.47) and Figure 8 shows that if $\bar{\Delta}_{n-1}(0) > 0$, then $\frac{d}{dr_{n-2}} \bar{\Delta}_{n-1}(0) > 0$ and $\frac{d}{dr_{n-2}} \bar{\Delta}_{n-1}(r_{n-2}) > 0$. This insures

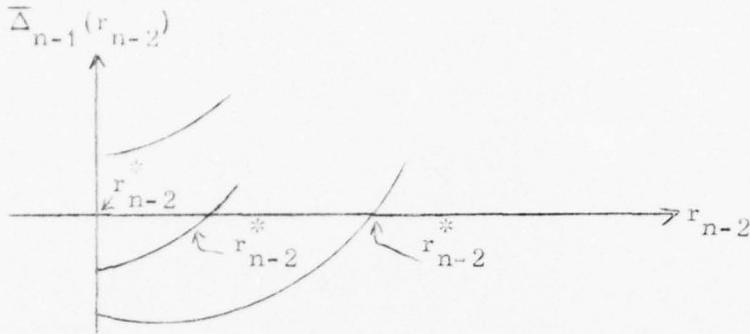


Figure 8:
Possible Convex Shapes for $\overline{\Delta}_{n-1}(r_{n-2})$

$\overline{\Delta}_{n-1}$ crosses the r_{n-2} -axis at most once. Therefore:

$$\exists \quad r_{n-2}^* \geq 0 \Rightarrow \overline{\Delta}_{n-1}(r_{n-2}) > 0 \text{ iff } r_{n-2} > r_{n-2}^*, \quad (4.48)$$

and:

$$\mathcal{C}_{n-1}(\alpha) = \{r_{n-2} : r_{n-2} < r_{n-2}^*\}, \quad (4.49)$$

Considering the decision at next earlier step, i.e., upon entering state $(n-2)$, the total reward until the end of the cycle is the sum of the expected reward during r_{n-2} and the expected reward after next transition. In a manner parallel to the development of (4.46), we get:

$$\begin{aligned} \Delta_{n-2}(\alpha) &= \mathcal{L}_{n-2}(\alpha, \bar{R}) - \mathcal{L}_{n-2}(\alpha, R) = E\{\beta_{n-2} r_{n-2}^2 - \alpha r_{n-2} | r_{n-3}\} \\ &\quad + \int_{r_{n-2}}^{\infty} \Delta_{n-1}(r_{n-2}) f(r_{n-2} | r_{n-3}) dr_{n-2} \\ &\stackrel{\Delta}{=} \overline{\Delta}_{n-2}(r_{n-3}, \beta_{n-2}, \alpha) \\ &\quad + \int_0^{\infty} \Delta_{n-1}(r_{n-2}) U(r_{n-2} - r_{n-2}^*) f(r_{n-2} | r_{n-3}) dr_{n-2} \\ &\stackrel{\Delta}{=} \overline{\Delta}_{n-2}(r_{n-3}, \beta_{n-2}, \alpha) + I_{n-2}(r_{n-3}) \quad . \end{aligned} \quad (4.50)$$

The first term on the right hand side of (4.50) is written in terms of the similar quadratic function defined in (4.47) except for different parameters. Therefore it has the same possible convex shapes shown in Figure 8. The second term $I_{n-2}(r_{n-3})$ on the right hand side of (4.50) is increasing in r_{n-3} because of Theorem (4.3) and that:

$$\Delta_{n-1}(r_{n-2}) U(r_{n-2} - r_{n-2}^*) , \quad (4.51)$$

is an increasing function of r_{n-2} . Furthermore, it is also quadratically increasing.

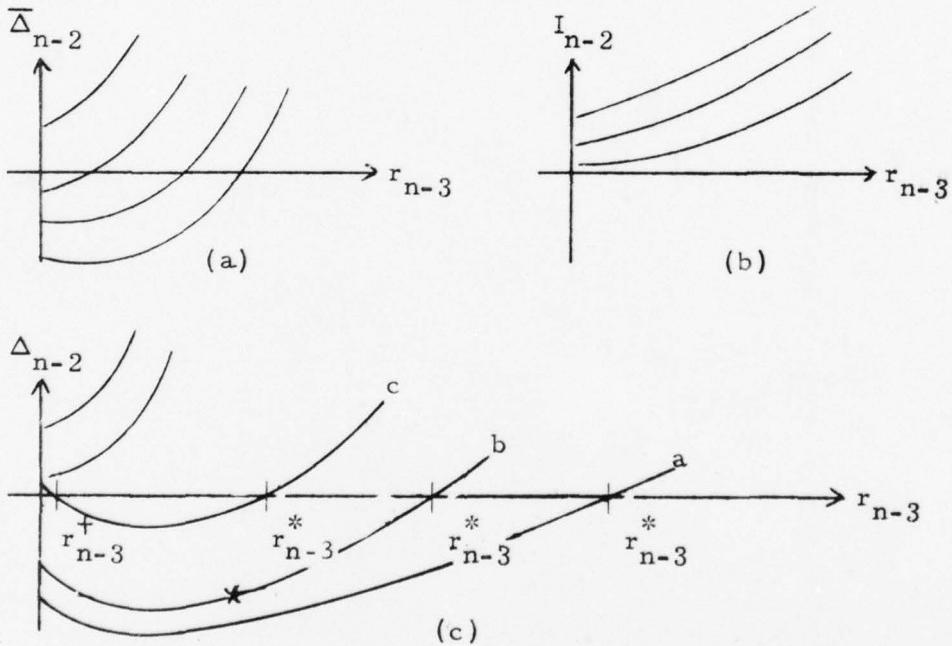


Figure 9:

The Sum of a Quadratic Function and a Convex Increasing Positive Function.

This resulting conditional expectation $I_{n-2}(r_{n-3})$ is also convex in r_{n-3} , because:

$$\Delta_{n-1}(r_{n-2}) U(r_{n-2} - r_{n-2}^*) ,$$

is a convex function of r_{n-2} and Theorem (4.10).

As a sum of a quadratic convex function and a positive increasing convex function, $\Delta_{n-2}(r_{n-3})$ has the possible convex shapes as shown in Figure 9(c). The curves a and b of Figure 9(c) have single crossings on r_{n-3} will. It is not clear whether curve c which has double crossings on r_{n-3} will happen or not.¹ If this does occur, we shall choose r_{n-3}^* as the crossing on the right, therefore:

$$\Delta_{n-2}(r_{n-3}) > 0 \quad \text{if} \quad r_{n-3} > r_{n-3}^*. \quad (4.52)$$

Because of the convexity of $\Delta_{n-2}(r_{n-3})$, there are at most two zero crossings on r_{n-3} , namely, r_{n-3}^+ and r_{n-3}^* . The global optimal region for replacement is:

$$\mathcal{C}_{n-2}(\alpha) = \{r_{n-3}: r_{n-3}^+ < r_{n-3} < r_{n-3}^*\}. \quad (4.53)$$

But the interval $\{r_{n-3}: 0 < r_{n-3} < r_{n-3}^*\}$ is locally optimal. Because for $|\epsilon| \rightarrow 0$:

$$\begin{aligned} & \int_{r_{n-3}^*}^{\infty} \Delta_{n-2}(r_{n-3}) f(r_{n-3} | r_{n-4}) dr_{n-3} \\ & > \int_{r_{n-3}^* + |\epsilon|}^{\infty} \Delta_{n-2}(r_{n-3}) f(r_{n-3} | r_{n-4}) dr_{n-3}. \end{aligned} \quad (4.54)$$

Because of the particular dependence relation among the r_i 's, the following conjecture concerning the single-zero crossing of Δ_{n-2} seems to be true¹:

¹The conjecture is later found to be correct. The proof and an extensive study are given in Appendix 1.

Conjecture: There exists r_{n-3}^* such that:

$$\Delta_{n-2}(r_{n-3}) > 0 \quad \text{iff} \quad r_{n-3} > r_{n-3}^*,$$

and :

$$\mathcal{C}_{n-2}(\alpha) = \{r_{n-3} : r_{n-3} < r_{n-3}^*\}.$$

Such nice analytical properties have eluded us at this time. However, numerical examples have all demonstrated that such simple threshold criteria for the optimal decisions are true. To consider entering states $(n-3), (n-4), \dots, 1, 0$, we have only to repeat our argument shown above. Hence we obtain the general recursive equation for the decision function on entering state $(n-i)$:

$$\begin{aligned} \Delta_{n-i}(r_{n-i-1}, \beta_{n-i}, \alpha) &= \bar{\Delta}_{n-i}(r_{n-i-1}, \beta_{n-i}, \alpha) \\ &+ \int_0^\infty \Delta_{n-i+1}(r_{n-i}, \beta_{n-i+1}, \alpha) U(r_{n-i} - r_{n-i}^*) f(r_{n-i} | r_{n-i-1}) dr_{n-i} \\ &\stackrel{\Delta}{=} \bar{\Delta}_{n-i}(r_{n-i-1}, \beta_{n-i}, \alpha) + I_{n-i}(r_{n-i-1}, \beta_{n-i+1}, \alpha). \end{aligned} \quad (4.55)$$

Here we have used the same conjecture² for $(n-i)$, i.e.:

$$\exists \quad r_{n-i-1}^* \geq 0 \ni \Delta_{n-i}(r_{n-i-1}) > 0 \quad \text{iff} \quad r_{n-i-1} > r_{n-i-1}^*,$$

and :

$$\mathcal{C}_{n-i}(\alpha) = \{r_{n-i-1} : r_{n-i-1} < r_{n-i-1}^*\}. \quad (4.56)$$

The maximal value of $\mathcal{L}_0(\alpha)$, for fixed α , is obtained by finding recursively the r_i^* 's ($i = 0, 1, \dots, n-2$) which maximize $\mathcal{L}_0(\alpha)$. Since all the r_j^* 's for $j > i$ are known when $\Delta_{i+1}(r_i, \beta_{i+1}, \alpha)$ is to be maximized, r_i^* is obtained by searching a zero crossing of a nonlinear function of a single variable. Thus the model is solved by the following algorithm:

² See Appendix 1.

(See Appendix 2 for the computational aspects.)

- a. Guess an initial value for α .
- b. Find $\mathcal{L}_0^0(\alpha)$, r_i^* ($i = 0, 1, \dots, n-2$) by solving recursive equations (4.55).
- c. If $|\mathcal{L}_0^0(\alpha)| < \epsilon \rightarrow 0$, stop computations (let $\alpha^* = \alpha$), otherwise, estimate a new value for α and return to b.

The following properties of $\mathcal{L}_0^0(\alpha)$ can be used to generate a sequence of α -values converging to one which satisfies the inequality in c.

i) $\mathcal{L}_0^0(\alpha)$ is monotone decreasing, since $\mathcal{L}_0(\alpha)$ has this property for a fixed policy (See Eq. (4.42)); and if $\mathcal{L}_0^0(\alpha_2) \geq \mathcal{L}_0^0(\alpha_1)$ for $\alpha_2 > \alpha_1$, then the policy used to achieve $\mathcal{L}_0^0(\alpha_2)$ could be used to achieve an $\mathcal{L}_0^0(\alpha_1) > \mathcal{L}_0^0(\alpha_1)$ -- a contradiction.

ii) As shown in the numerical example below (page 58), $\alpha^*(\rho=0)$ is relatively easy to compute. This value is a lower bound on $\alpha^*(\rho \neq 0)$.

Appendix 3 shows the flow chart of the algorithm.

Examples (Linear Reward Rate Case): Tables 3 and 4 present results from some numerical examples based on the models in this section. Figures 10 and 11 show reward and policies for different values of correlation. Assuming that $p = 5.0$, $d = 1$, $\eta_i = 2.0$, $n = 5$, $\beta_0 = 5$, $\beta_1 = 4$, $\beta_2 = 3$, $\beta_3 = 2$, $\beta_4 = 1$ and $\beta_5 = 0$. The results indicate that the optimal average reward per unit time increases as the correlation increases. We present details of the computation involved. The recursive function (Equation (4.55)) is:

$$\Delta_i(r_{i-1}, \beta_i, \alpha) = \bar{\Delta}_i(r_{i-1}, \beta_i, \alpha) + \int_{r_i^*}^{\infty} \Delta_{i+1}(r_i, \beta_{i+1}, \alpha) f(r_i | r_{i-1}) dr_i .$$

We list the first term on the right hand side of the above equation:

$$\begin{aligned}\overline{\Delta}_i(r_{i-1}, \beta_i, \alpha) &= E\{\beta_i r_i^2 - \alpha r_i | r_{i-1}\} \\ &= \beta_i \rho^2 r_{i-1}^2 + \{\beta_i 8\rho(1-\rho) - \alpha\rho\} r_i + \{\beta_i 8(1-\rho)^2 - 2\alpha(1-\rho)\} ,\end{aligned}\quad (4.57)$$

for different i:

$$\left\{ \begin{array}{l} \overline{\Delta}_4(r_3, 1, \alpha, \rho) = \rho^2 r_3^2 + \{8\rho(1-\rho)-\alpha\rho\} r_3 + 8(1-\rho)^2 - 2\alpha(1-\rho) , \\ \overline{\Delta}_3(r_2, 2, \alpha, \rho) = 2\rho^2 r_3^2 + \{16\rho(1-\rho)-\alpha\rho\} r_3 + 16(1-\rho)^2 - 2\alpha(1-\rho) , \\ \overline{\Delta}_2(r_1, 3, \alpha, \rho) = 3\rho^2 r_1^2 + \{24\rho(1-\rho)-\alpha\rho\} r_1 + 24(1-\rho)^2 - 2\alpha(1-\rho) , \\ \overline{\Delta}_1(r_0, 4, \alpha, \rho) = 4\rho^2 r_0^2 + \{32\rho(1-\rho)-\alpha\rho\} r_0 + 32(1-\rho)^2 - 2\alpha(1-\rho) , \\ \overline{\Delta}_0(r_{-1} = 0, 5, \alpha, \rho = 0) = 40 - 2\alpha . \end{array} \right.$$

The overall Δ_i will be the sum of $\overline{\Delta}_i$ and the repeated integral I_i . The optimal $\mathcal{L}_0(\alpha)$ for fixed value of α is:

$$\mathcal{L}_0(\alpha) = -p - \alpha d + \overline{\Delta}_0(\alpha) + \int_{r_0}^{\infty} \Delta_1(r_0) f(r_0) d r_0 , \quad (4.58)$$

where $f(r_0)$ is an exponential density function with the mean value equal to 2.

In the following we study the two extremal cases, i.e., $\rho = 1$ and $\rho = 0$. These cases could be done by analysis with minimal computer usage.

Case 1 ($\rho = 1$): There is only one random variable, r_0 , in this case with $r_1 = r_2 = r_3 = r_4 = r_0$. The optimal replacement state N will be a function of r_0 . The equivalent optimal average reward rate can be represented as:

$$\mathcal{L}_0(\alpha) = E\{(\beta_0 + \beta_1 + \dots + \beta_{N-1})r_0^2 - p - \alpha(Nr_0 + d)\} . \quad (4.59)$$

If $\beta_N r_0^2 - \alpha r_0 < 0$, we replace on entering state N. Therefore, we replace on entering state N if r_0 falls in the given interval:

r_0	N
$0 \leq r_0 < \frac{\alpha}{\beta_1}$	1
$\frac{\alpha}{\beta_1} \leq r_0 < \frac{\alpha}{\beta_2}$	2
$\frac{\alpha}{\beta_2} \leq r_0 < \frac{\alpha}{\beta_3}$	3
$\frac{\alpha}{\beta_3} \leq r_0 < \frac{\alpha}{\beta_4}$	4
$\frac{\alpha}{\beta_4} \leq r_0 < \frac{\alpha}{\beta_5} = \infty$	5

Then, we have :

$$\begin{aligned}
 \mathcal{L}_0(\alpha) &= -p - \alpha d + \int_0^{\frac{\alpha}{\beta_1}} (\beta_0 r_0^2 - \alpha r_0) \frac{1}{\eta} e^{-\frac{r_0}{\eta}} dr_0 + \int_{\frac{\alpha}{\beta_1}}^{\infty} (\beta_1 r_0^2 - \alpha r_0) \frac{1}{\eta} e^{-\frac{r_0}{\eta}} dr_0 \\
 &\quad + \int_{\frac{\alpha}{\beta_2}}^{\infty} (\beta_2 r_0^2 - \alpha r_0) \frac{1}{\eta} e^{-\frac{r_0}{\eta}} dr_0 + \int_{\frac{\alpha}{\beta_3}}^{\infty} (\beta_3 r_0^2 - \alpha r_0) \frac{1}{\eta} e^{-\frac{r_0}{\eta}} dr_0 \\
 &\quad + \int_{\frac{\alpha}{\beta_4}}^{\infty} (\beta_4 r_0^2 - \alpha r_0) \frac{1}{\eta} e^{-\frac{r_0}{\eta}} dr_0 \\
 &= -5 - \alpha + 40 - 2\alpha + 2\alpha(e^{-\frac{\alpha}{8}} + e^{-\frac{\alpha}{6}} + e^{-\frac{\alpha}{4}} + e^{-\frac{\alpha}{2}}) + 32e^{-\frac{\alpha}{8}} + 24e^{-\frac{\alpha}{6}} + 16e^{-\frac{\alpha}{4}} + 8e^{-\frac{\alpha}{2}} . \quad (4.60)
 \end{aligned}$$

The optimal α^* which satisfies $\mathcal{L}_0^0(\alpha^*) = 0$ is easy to obtain and equals 16.10. The optimal thresholds $\{r_i^* = \frac{\alpha^*}{\beta_{i+1}}, i = 0, 1, 2, 3\}$ are obtained from α^* .

Case 2 ($\rho = 0$): There is no dependency on the past. Therefore the optimal policy is to find a fixed strategy such as a critical state k^* . We definitely replace it on entering state k^* . To represent α^* as a function of ρ :

$$\alpha^*(\rho) = \frac{E \sum_0^{N-1} \beta_i r_i^2 - p}{E \sum_0^{N-1} r_i + d}, \quad (4.61)$$

where N is a random variable. For the fixed k^* , we have:

$$\alpha^*(\rho=0) = \frac{E \sum_0^{k^*-1} \beta_i r_i^2 - p}{E \sum_0^{k^*-1} r_i + d}. \quad (4.62)$$

If we use the same strategy for the case when $\rho \neq 0$, then it is easily seen that the average reward per unit time is $\alpha^*(\rho = 0)$. As the optimal strategy should not work worse than the above strategy which is optimal for $\rho = 0$, we have:

$$\alpha^*(\rho) \geq \alpha^*(\rho = 0). \quad (4.63)$$

With our replacement policy structure, the independent case will have the minimum average reward per unit time. To find the k^* for the independent case, we have only to look for the maximum of the average reward rate which is a function of k .

k-Fixed Replacement State	Average Reward Rates
	$\frac{E(\sum_{i=1}^{\beta-1} \beta_i r_i^2) - p}{\sum_{i=0}^{k-1} r_i + d}$
1	$\frac{5 \times 8 - 5}{2+1} = 11\frac{2}{3}$
2	$\frac{(5+4)8-5}{2+2+1} = 13.4$
3	$\frac{(5+4+3)8-5}{2+2+2+1} = 13.0$

Therefore, $k^* = 2$ and we decide to replace the part on entering state 2.

TABLE 3			
Linear Reward Rate Case: The Fast Convergence of $(\alpha, \mathcal{L}_0(\alpha))$ to $(\alpha^*, \mathcal{L}_0^0(\alpha^*))$. Use the Linear Interpolation Method.			
$p = 5, d = 1, \eta_i = 2, n = 5, \beta_0 = 5, \beta_1 = 4, \beta_2 = 3, \beta_3 = 2, \beta_4 = 1$ and $\beta_5 = 0$.			
ρ	First Iteration	Second Iteration	Third Iteration
0.0	(13.40, 0.0)		
0.25	(13.82, -1.098183)	(13.70, -0.528967)	(13.59, -0.001812)
0.50	(13.00, 5.802)	(15.00, -3.375)	(14.25, -0.0688)
0.75	(14.96, 0.115082)	(15.04, -0.237404)	(14.985, 0.004868)
1.00	(16.10, -0.005)		

TABLE 4 Linear Reward Rate Case: The Optimal Replacement Thresholds.				
$p = 5, d = 1, \eta_i = 2, n = 5, \beta_0 = 5, \beta_1 = 4, \beta_2 = 3, \beta_3 = 2, \beta_4 = 1, \text{ and } \beta_5 = 0.$				
ρ	r_0^*	r_1^*	r_2^*	r_3^*
0.0	0.0	$+\infty$	$+\infty$	$+\infty$
0.25	0.7	3.8	11.3	37.0
0.50	1.9	4.1	8.7	22.5
0.75	3.0	4.7	8.1	17.9
1.00	4.25	5.36	8.05	16.10

4.5 Constant Reward Rate - After Set-up (Alignment) Time

In practice, the time of set-up after each transition occurs is finite, and this time should be included in the analysis of the model. In fact, the inclusion of the set-up time generally causes major changes in the results.

In general, the set-up time will be denoted by c . To begin with, the reward rates after the set-up period will each be assumed to be constant. For simplicity, we also assume that all $p_i = p$ and all $d_i = d$. Similar to the constant reward case, the decision rules for the severely deteriorated states are to replace it definitely. Therefore we can find an optimal critical state k^* . By utilizing the stochastically increasing property of $f(r_i | r_{i-1})$, the optimal rules on entering state $\{i : i < k^*\}$ will depend on the observed duration $\{r_j, 0 \leq j \leq i-1\}$. The optimal threshold $\{r_i^* : k^*-1 \leq i < n-2\}$ will be infinite. We definitely replace

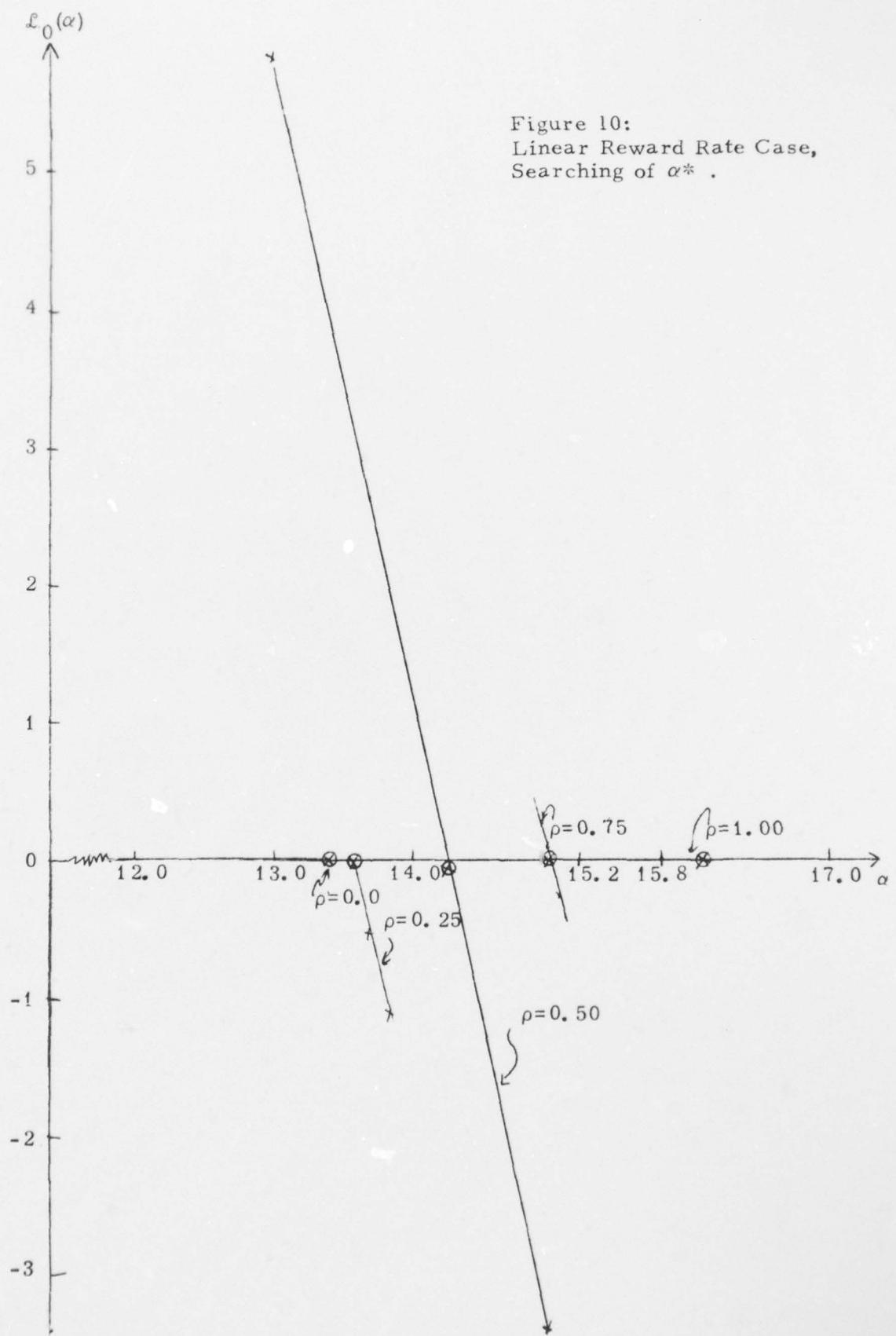
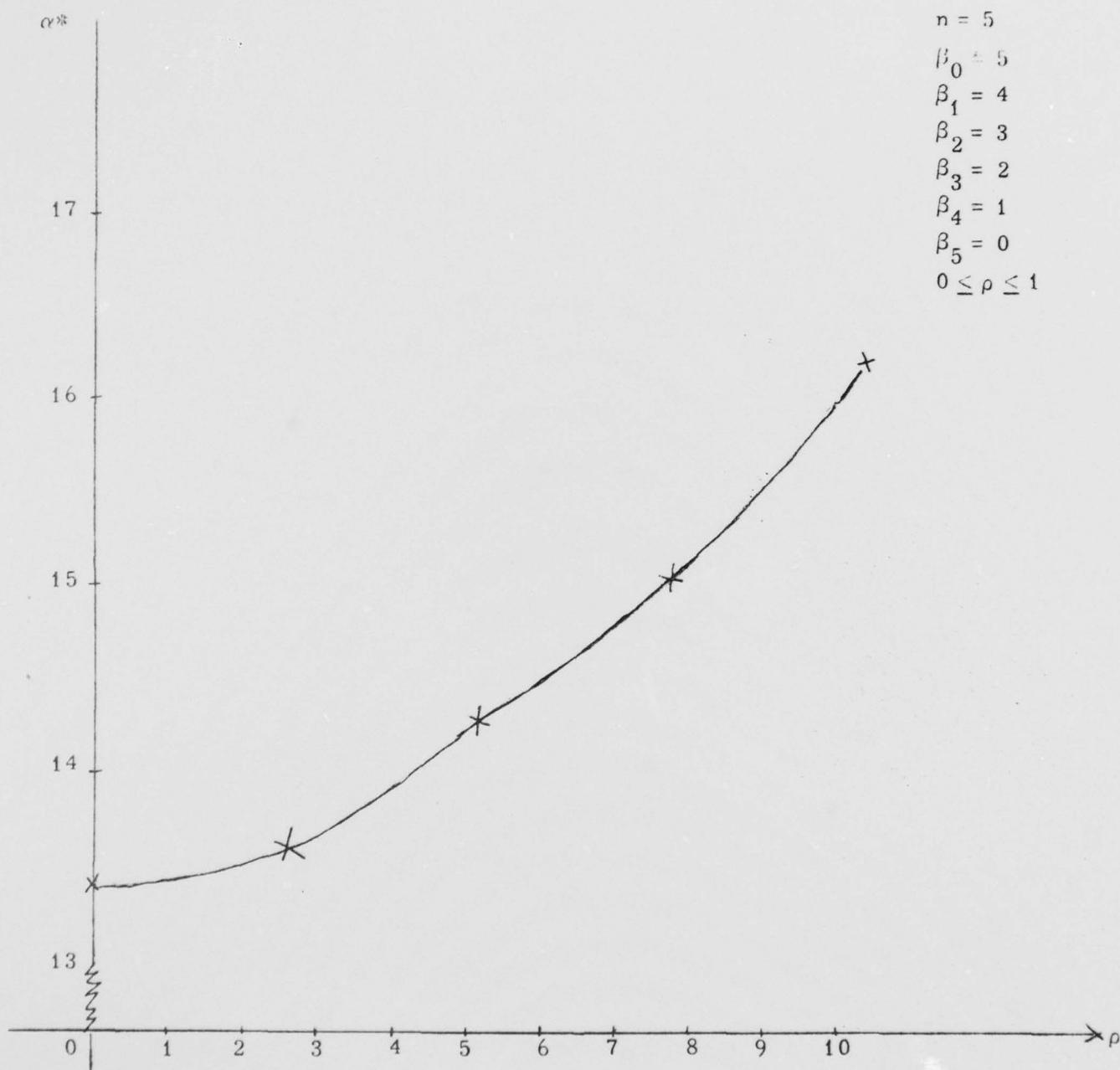


Figure 10:
Linear Reward Rate Case,
Searching of α^* .



it on entering state k^* .

In this section we will combine the techniques used in Section 4.3 and Section 4.4.

Based on the problem statement in Section 4.1, the optimal decision on entering state j must maximize the mean future reward until the next renewal (the Principal of Optimality), i.e., $\mathcal{L}_j(\alpha)$, for a suitable α , we have:

$$\mathcal{L}_j(\alpha) = E \left[\sum_{i=j}^{N-1} \beta_i(r_i - c) u(r_i - c) | r_{j-1} \right] - p - \alpha E \left[\sum_{i=j}^{N-1} r_i | r_{j-1} \right] - \alpha d . \quad (4.64)$$

The optimal decisions on entering each state will be found in terms of α , and then the proper α^* (for producing decisions which maximize L) is the one for which the maximum:

$$\mathcal{L}_0^0(\alpha^*) = 0 .$$

The reward structure during each state (r_i) for this case has the form:

$$E[c_i(r_i) | r_{i-1}] = 0 \cdot P[r_i < c | r_{i-1}] \quad (4.65) \\ + \beta_i(-c + E[r_i | r_{i-1}, r_i > c]) \cdot P(r_i > c | r_{i-1}) .$$

Optimization by dynamic programming begins with considering the decision at the last step, i.e., when the system enters state $(n-1)$. There are two choices, to replace (R) or not to replace (\bar{R}) , with corresponding values:

$$\mathcal{L}_{n-1}(\alpha; R) = -p - \alpha d , \quad (4.66)$$

$$\mathcal{L}_{n-1}(\alpha; \bar{R}) = \beta_{n-1}(-c + E[r_{n-1} | r_{n-2}, r_{n-1} > c]) \cdot P(r_{n-1} > c | r_{n-2}) \\ - \alpha E[r_{n-1} | r_{n-2}] - p - \alpha d , \quad (4.67)$$

for $f_{n-1}(\alpha)$. Consider the difference:

$$\begin{aligned}
 \Delta_{n-1}(r_{n-2}, \beta_{n-1}, \alpha) &= \mathcal{L}_{n-1}(\alpha; \bar{R}) - \mathcal{L}_{n-1}(\alpha; R) \\
 &= \int_0^c (-\alpha) r_{n-1} f(r_{n-1} | r_{n-2}) dr_{n-1} \\
 &\quad + \int_c^\infty (\beta_{n-1}(r_{n-1}-c) - \alpha r_{n-1}) f(r_{n-1} | r_{n-2}) dr_{n-1} \\
 &= \int_0^\infty (-\alpha) r_{n-1} f(r_{n-1} | r_{n-2}) dr_{n-1} \\
 &\quad + \int_c^\infty \beta_{n-1}(r_{n-1}-c) f(r_{n-1} | r_{n-2}) dr_{n-1} \\
 &= \int_0^\infty (-\alpha) r_{n-1} f(r_{n-1} | r_{n-2}) dr_{n-1} \\
 &\quad + \int_c^\infty \beta_{n-1} r_{n-1} f(r_{n-1} | r_{n-2}) dr_{n-1} \\
 &\quad - \beta_{n-1} c \int_c^\infty f(r_{n-1} | r_{n-2}) dr_{n-1} \\
 &= \int_0^\infty (-\alpha) r_{n-1} f(r_{n-1} | r_{n-2}) dr_{n-1} \\
 &\quad + \int_0^\infty \beta_{n-1} r_{n-1} f(r_{n-1} | r_{n-2}) dr_{n-1} \\
 &\quad - \int_0^c \beta_{n-1} r_{n-1} f(r_{n-1} | r_{n-2}) dr_{n-1} \\
 &\quad - \beta_{n-1} c \int_c^\infty f(r_{n-1} | r_{n-2}) dr_{n-1} \\
 &= (\beta_{n-1} - \alpha) \int_0^\infty r_{n-1} f(r_{n-1} | r_{n-2}) dr_{n-1}
 \end{aligned}$$

$$\begin{aligned}
 & - \int_0^c \beta_{n-1} r_{n-1} f(r_{n-1} | r_{n-2}) dr_{n-1} \\
 & - \beta_{n-1} c \int_c^\infty f(r_{n-1} | r_{n-2}) dr_{n-1} \\
 & \triangleq \bar{\Delta}_{n-1}(r_{n-2}, \beta_{n-1}, \alpha) \quad . \quad (4.68)
 \end{aligned}$$

The $\bar{\Delta}_i$, for $i = 0, \dots, n-1$, will have a form similar to that for (n-1) except that β_{n-1} is replaced with β_i and:

$$\bar{\Delta}_i(r_{i-1}, \beta_i, \alpha) = \int_0^\infty G_i(r_i) f(r_i | r_{i-1}) dr_i \quad . \quad (4.69)$$

The integrand G_i of $\bar{\Delta}_i$ has the possible forms as shown in Figure 12.

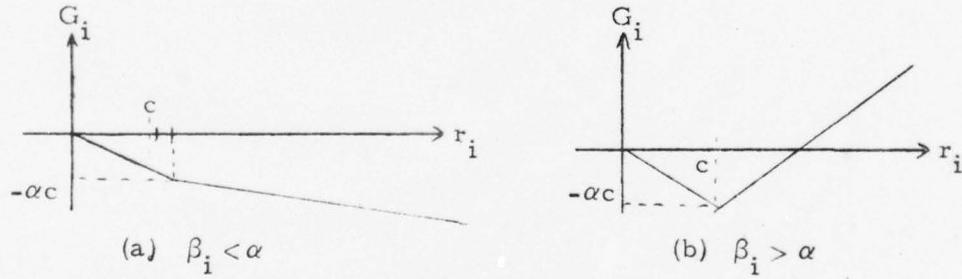


Figure 12:

Possible Shapes for the Integrand G_i .

Going back to Eq. (4.69), the decision function will always be negative if $\beta_{n-1} < \alpha$, which requires replacement on entering state (n-1) if the system reaches that state. Next we consider the decision on entering (n-2) with the assumption that $\beta_{n-1} < \alpha$, then Δ_{n-2} is :

$$\begin{aligned}
 \Delta_{n-2}(r_{n-3}, \beta_{n-2}, \alpha) &= (\beta_{n-2} - \alpha) \int_0^\infty r_{n-2} f(r_{n-2} | r_{n-3}) dr_{n-2} \\
 &\quad - \beta_{n-2} \int_0^\infty [\min(c, r_{n-2})] f(r_{n-2} | r_{n-3}) dr_{n-2} \\
 &\stackrel{\Delta}{=} \bar{\Delta}_{n-3}(r_{n-3}, \beta_{n-2}, \alpha)
 \end{aligned} \tag{4.70}$$

$$\bar{\Delta}_{n-2}(r_{n-3}) < 0 \quad \text{for all } r_{n-3} \quad \text{if} \quad \beta_{n-2} < \alpha.$$

This iteration may eventually reach a state $(k-1)$ such that $\beta_{k-1} > \alpha$, where $(k-1) \in (0, 1, 2, \dots, n-2)$. We have:

$$\begin{aligned}
 \Delta_{k-1}(r_{k-2}, \beta_{k-1}, \alpha) &= \bar{\Delta}_{k-1}(r_{k-2}, \beta_{k-1}, \alpha) \\
 &\quad + \{\text{Future Reward Before Renewal} = 0\} \\
 &= (\beta_{k-1} - \alpha) \int_0^\infty r_{k-1} f(r_{k-1} | r_{k-2}) dr_{k-1} \\
 &\quad - \beta_{k-1} \int_0^\infty [\min(c, r_{k-1})] f(r_{k-1} | r_{k-2}) dr_{k-1} \\
 &= \int_0^\infty G_{k-1}(r_{k-1}) f(r_{k-1} | r_{k-2}) dr_{k-1}.
 \end{aligned} \tag{4.71}$$

The integrand $G_{k-1}(r_{k-1})$ has the shape as shown in Figure 12(b), which is convex on r_{k-1} and increasing when $r_{k-1} > c$. Furthermore, $\bar{\Delta}_{k-1}(r_{k-2}, \beta_{k-1}, \alpha)$ is convex on r_{k-2} from Theorem (4.10).

It is easy to see that $\Delta_{k-1}(r_{k-2})$ is an increasing function for large r_{k-2} . We pose a conjecture similar to the one in Section 4.3.

Conjecture: There exists r_{k-2}^* such that:

$$\Delta_{k-1}(r_{k-2}) > 0 \quad \text{iff} \quad r_{k-2} > r_{k-2}^*,$$

and the optimal decision region³:

$$\mathcal{C}_{k-1}(\alpha) = \{r_{k-2} : r_{k-2} < r_{k-2}^*\} .$$

We now consider the decision at its previous state, i.e., state $(k-2)$, the total reward until the end of the cycle is the sum of expected reward during r_{k-2} and expected reward after next transition. In a manner parallel to the development of (4.70), we obtain:

$$\begin{aligned}\Delta_{k-2}(r_{k-3}, \beta_{k-2}, \alpha) &= \bar{\Delta}_{k-2}(r_{k-3}, \beta_{k-2}, \alpha) \\ &+ \int_{r_{k-2}}^{\infty} \Delta_{k-2}, \beta_{k-1}, \alpha f(r_{k-2} | r_{k-3}) dr_{k-2} \\ &= \bar{\Delta}_{k-2}(r_{k-3}) + I_{k-2}(r_{k-3}) ,\end{aligned}\quad (4.72)$$

where $\bar{\Delta}_{k-2}$ is convex on all r_{k-3} and increasing over large values of r_{k-3} . The second term (I_{k-2}) on the right hand side of Equation (4.72) is stochastically increasing in r_{k-3} and convex on r_{k-3} because:

$$\Delta_{k-1}(r_{k-2}) U(r_{k-2} - r_{k-2}^*) ,$$

is an increasing and convex function of r_{k-2} from Theorem (4.3) and Theorem (4.10).

As a sum of a convex function and a positive increasing convex function, $\Delta_{k-3}(r_{k-3})$ is also a convex function of r_{k-3} . We use the same conjecture³:

$$\exists r_{k-3}^* > 0 \ni \Delta_{k-2}(r_{k-3}^*) > 0 \text{ iff } r_{k-3} > r_{k-3}^* ,$$

and:

³See Appendix 1.

$$\mathcal{C}_{k-2}(\alpha) = \{r_{k-3}: r_{k-3} < r_{k-3}^*\} .$$

When considering entering states $(k-3), (k-4), \dots, (k-i), \dots, 1$, we repeat our argument shown above. We form the general recursive equation for the decision function on entering state $(k-i)$:

$$\begin{aligned} \Delta_{k-i}(r_{k-i-1}, \beta_{k-i}, \alpha) &= \bar{\Delta}_{k-i}(r_{k-i-1}, \beta_{k-i}, \alpha) \\ &+ \int_{r_{k-i}}^{\infty} \Delta_{k-i+1}(r_{k-i}, \beta_{k-i+1}, \alpha) f(r_{k-i} | r_{k-i-1}) dr_{k-i}. \end{aligned} \quad (4.73)$$

Here we use the same conjecture for $k-i$, i.e.:

$$\exists \quad r_{k-i-1}^* > 0 \quad \Delta_{k-i}(r_{k-i-1}) > 0 \quad \text{if} \quad r_{k-i-1} > r_{k-i-1}^* ,$$

and :

$$\mathcal{C}_{k-i}(\alpha) = \{r_{k-i-1}: r_{k-i-1} < r_{k-i-1}^*\} .$$

The maximal value of $\mathcal{L}_0(\alpha)$, for fixed α , is obtained by finding recursively the r_i^* ($i = 0, 1, \dots, k, \dots, n-2$) which maximize $\mathcal{L}_0(\alpha)$. Since all the r_j^* 's for $j > i$ are known when Δ_{i+1} is to be maximized, r_i^* is obtained by searching a zero crossing of a nonlinear function of a single variable. Thus, the model is solved by the following algorithm: (see Appendix 4 for the flow chart)

- a. Guess an initial value for α (possibly α^* for constant reward case with the same reward structure).
- b. Find k , such that $\beta_i < \alpha$ ($i = k, \dots, n-1$). Let $r_i^* = \infty$ ($i = k-1, \dots, n-2$).
- c. Find $\mathcal{L}_0^0(\alpha) = \max\{\mathcal{L}\}$, r_i^* ($i = 0, 1, \dots, k-1$) by solving recursive equations (4.73).

- d. If $|\mathcal{L}_0^0(\alpha)| < \epsilon \rightarrow 0$, stop computations, (let $k^* = k$, $\alpha^* = \alpha$)
otherwise, estimate a new value for α and return to b.

4.6 Conclusions

The optimal replacement of deteriorating parts based on the deterioration process described by a multivariate exponential distribution was examined. For the constant reward rate case, the optimization procedure does not benefit from the history of deterioration. However, we do utilize this knowledge to optimize the linear reward rate case and the constant reward rate after set-up time case.

Numerical examples show that the average reward per unit time for certain reward structures is an increasing function of the correlation ρ . Due to our assumption that the transition times are directly monitored, the prediction of the future after each transition should depend on the history and the correlation between the past and the future. The theory we suggest is that if the correlation increases, then the certainty of the future also increases. Therefore, the optimal policy is more efficient as the certainty of the future increases.

CHAPTER V

CONCLUSIONS

Multivariate exponential distributions have been introduced into both downtime modeling and deterioration modeling. The one based on the sum of the squares of multinormal variables is selected here, because of its physical motivation and analytical simplicity.

The sums of multivariate exponential variables are found to well approximate the frequently used lognormal downtime distribution, except for the third and fourth moments. If the subsidiary times of a downtime can be characterized as exponential variables, then the lognormal distribution, to our judgment, is used just for the purpose of expediency.

Several dependence properties are developed for this particular kind of distribution. The monotonicity-preserving transformation was applied by Barlow and Proschan[2] to the reliability theory. Here we have introduced the convexity-preserving transformation to problems in the same area.

Different reward structures such as constant, linear, and set-up cases are considered for deterioration modeling. Only incomplete results have been found concerning the conjecture that there is at most one zero crossing of the decision function. However, we have shown that there are at most two zero crossings. Also the numerical examples have all demonstrated that we have only single decision threshold.

Areas for further research include both extensions of our results to gamma marginal distributions and nonstationary correlations.

It could also be interesting to consider systems of more than one component, i. e., correlations exist both in parallel and in series.

The scheduling of inspections for systems in which degree of deterioration can be observed only through inspections is difficult but challenging.

We modeled downtime and deterioration separately. It is interesting to consider them together. The time to carry out replacement of the deterioration modeling can be a downtime process.

APPENDIX 1

PROOF OF THE ZERO-CROSSING THEOREM

The conjecture concerning the zero-crossing properties of the decision functions in Chapter 4 is proved here in a more general form. Let the durations in states $0, 1, 2, \dots, n-1$ be denoted by r_0, r_1, \dots, r_{n-1} and assume that $r_i, 0 \leq i \leq n-1$, are a Markovian sequence with stationary probability transition function:

$$f_{r_i | r_{i-1}}(y|x) = f_{r_1 | r_0}(y|x) = f(y|x)$$

for all $2 \leq i \leq n-1$, where $f_{r_1 | r_0}(y|x)$ is given in Equation (4.17); note that η_1 is not necessarily equal to η_0 . As we have already shown this conditional density function is TP_∞ and it preserves convexity when used as a kernel in an integral transformation. We also assume that at state i the replacement cost and time, i.e., p_i and d_i are non negative random variables independent of $r_j, 0 \leq j \leq i-1$. The reward rate at state i is again assumed to be equal to $\beta_i c(t)$ where $c(t)$ is a nonnegative function and $\beta_i \geq \beta_{i+1} \geq 0$ for $0 \leq i \leq n-1$, $\beta_n = 0$. The problem is to find the optimal stopping time N which, for a fixed $\alpha \geq 0$, maximizes:

$$E \left[\sum_{i=0}^{N-1} \beta_i \int_0^{r_i} c(t) dt - p_N - \alpha (d_N + \sum_{i=0}^{N-1} r_i) \right] , \quad (\text{A1.1})$$

where N depends only on $r_i, 0 \leq i \leq N-1$. Because of the Markov property, an equivalent problem is to find the optimal decision rule which requires that the part be replaced upon entering state j iff $r_{j-1} \in \mathcal{C}_j(\alpha)$. In this Appendix we give a sufficient condition under which $c_j(\alpha)$ has a very simple form, namely $\mathcal{C}_j(\alpha) = [0, r_{j-1}^*]$. We need the following definitions and

theorem about the variation-diminishing property of total positive kernels.

All the material is taken from Karlin [17] with only slight changes.

Let $h(t)$ be a function defined in $[0, \infty)$ and let:

$S^-(h) = S^-[h(t)] = \sup S^- [h(t_1), h(t_2), \dots, h(t_m)]$, $(t_i \in [0, \infty)$, m is arbitrary but finite, and $S^-(x_1, x_2, \dots, x_m)$ is the number of sign changes of the indicated sequence, zero terms being discarded.

Let $k(x, y)$ defined on $[0, \infty) \times [0, \infty)$ be Borel-measurable and assume that the integral $\int_0^\infty k(x, y) d\mu(y)$ exists for any x in $[0, \infty)$. Here μ represents a fixed sigma-finite regular measure defined on $[0, \infty)$ such that $\mu(U) > 0$ for each open set U for which $U \cap [0, \infty)$ is nonempty. Let h be bounded and Borel-measurable on $[0, \infty)$, and consider the transformation:

$$g(x) = (Th)(x) = \int_0^\infty k(x, y) h(y) d\mu(y)$$

Theorem (A1.1) (Theorem 3.1 on p. 21 of Karlin [17]):

If k is TP_r and satisfies the integrability requirements stated above, then:

$$S^-(g) = S^-(Tf) \leq S^-(h) \text{ provided } S^-(h) \leq r-1.$$

For the case in which h is piecewise-continuous, if $S^-(g) = S^-(h) \leq r-1$, we further assert that the values of the functions h and g exhibit the same sequence of signs when their respective arguments traverse the half line $[0, \infty)$ from left to right.

Theorem (A1.2) Let $h(y)$ be a Borel-measurable function such that

$$|h(y)| \leq a + by^{2m} \text{ for some } a > 0, b > 0, \text{ and positive integer } m. \text{ Let:}$$

$$g(x) = \int_0^\infty h(y) f(y|x) dy$$

where $f(y|x)$ is the conditional density function of r_i given r_{i-1} . Then:

(i) $S^-(g) \leq S^-(h)$,

(ii) there exists a' and b' such that $|g(x)| \leq a' + b'x^{2m}$.

Proof: We can write:

$$g(x) = \int_0^\infty [h(y) / (a + b y^{2m})] f(y|x) d\mu(y)$$

where:

$$d\mu(y) = (a + b y^{2m}) dy.$$

It is clear that:

$$\int_0^\infty f(y|x) d\mu(y),$$

exists for all $x \in [0, \infty)$ and that $h(y) / (a + b y^{2m})$ is bounded. Since $f(y|x)$ is TP_∞ , by the above theorem $S^-(g) \leq S^-(h)$. Next consider:

$$|g(x)| \leq \int_0^\infty |h(y)| f(y|x) dx \leq a + b \int_0^\infty y^{2m} f(y|x) dx,$$

but it is a well known fact that $\int_0^\infty y^{2m} f(y|x) dy$ is a polynomial of degree at most $2m$ [33], therefore (ii) is established.

Going back to our problem of (A1.1). If we assume the reward rate function $c(t)$ is a non-decreasing function not growing faster than a certain polynomial (e.g., a constant, a linear function in time or either of the two after set-up). Then:

$$w(t) \triangleq \int_0^t c(t) dt,$$

is a continuous convex function, $w(0) = 0$, and growing not faster than a certain polynomial. We shall also write $E(p_i + \alpha d_i) = e_i(\alpha)$ for $1 \leq i \leq n$.

By using the dynamic programming approach to solving this optimization problem we have the following:

$$\mathcal{L}_{n-1}(\alpha, R) = E(-p_{n-1} - \alpha d_{n-1} | r_{n-2}) = -e_{n-1}(\alpha) ,$$

$$\mathcal{L}_{n-1}(\alpha, \bar{R}) = E[\beta_{n-1} w(r_{n-1}) - \alpha r_{n-1} - p_n - \alpha d_n | r_{n-2}]$$

$$= E[\beta_{n-1} w(r_{n-1}) - \alpha r_{n-1} | r_{n-2}] - e_n(\alpha) ,$$

$$\Delta_{n-1}(\alpha) = \Delta_{n-1}(\alpha, r_{n-2}) = E[\beta_{n-1} w(r_{n-1}) - \alpha r_{n-1} | r_{n-2}] - e_n(\alpha) + e_{n-1}(\alpha) ,$$

and in general, $1 \leq j \leq n-2$:

$$\mathcal{L}_j(\alpha, R) = -e_j(\alpha) ,$$

$$\mathcal{L}_j(\alpha, \bar{R}) = E[\beta_j w(r_j) - \alpha r_j$$

$$+ \max(\mathcal{L}_{j+1}(\alpha, R), \mathcal{L}_{j+1}(\alpha, \bar{R})) | r_{j-1}]$$

$$= E[\beta_j w(r_j) - \alpha r_j - e_{j+1}(\alpha) + (\Delta_{j+1}(\alpha))^+ | r_{j-1}] ,$$

where for $1 \leq j \leq n-1$:

$$\Delta_j(\alpha) = \Delta_j(\alpha, r_{j-1}) = \mathcal{L}_j(\alpha, \bar{R}) - \mathcal{L}_j(\alpha, R)$$

$$= E[\beta_j w(r_j) - \alpha r_j + (\Delta_{j+1}(\alpha))^+ | r_{j-1}] + e_j(\alpha) - e_{j+1}(\alpha) ,$$

and $(x)^+ = \max(0, x)$. Then the optimal decision rule upon entering state j is: (i) to replace the part if $\Delta_j(\alpha, r_{j-1}) < 0$; (ii) not to replace if $\Delta_j(\alpha, r_{j-1}) > 0$; (iii) either one if $\Delta_j(\alpha, r_{j-1}) = 0$.

Theorem (A1.3): Let α be fixed, $\alpha \geq 0$. If the following assumption are satisfied:

(i) $\beta_i \geq \beta_{i+1}$, $0 \leq i \leq n-1$, $\beta_n = 0$,

(ii) $c(t)$ is non-decreasing and bounded above by a polynomial,

(iii) the sequence $e_j(\alpha)$, $1 \leq j \leq n$, is convex and non-decreasing, i.e.,

$e_j \leq e_{j+1}$ and $e_j \leq (e_{j-1} + e_{j+1}) / 2$, then there exist $0 \leq r_0^* \leq r_1^* \leq \dots \leq r_{n-1}^* \leq \infty$ such that the optimal decision rule upon entering state j is:

(i) to replace if $r_{j-1} < r_{j-1}^*$: and (ii) not to replace if $r_{j-1} \geq r_{j-1}^*$.

Proof: Starting with the $(n-1)$ th state, the difference in expected reward of replacing and not replacing the part given past observation is:

$$\Delta_{n-1}(\alpha) = E[G_{n-1}(\alpha, r_{n-1}) | r_{n-2}] ,$$

where:

$$G_{n-1}(\alpha, r_{n-1}) = \beta_{n-1} w(r_{n-1}) - \alpha r_{n-1} - e_n(\alpha) + e_{n-1}(\alpha) .$$

Since $w(r_{n-1})$ is a continuous convex function, thus $G_{n-1}(\alpha, r_{n-1})$ is also continuous, convex, bounded in absolute value by a polynomial, and by assumption (iii):

$$G_{n-1}(\alpha, 0) = e_{n-1}(\alpha) - e_n(\alpha) \leq 0 .$$

This implies that $G_{n-1}(\alpha, r)$ can have at most one sign change, i.e.:

$$S^-(G_{n-1}(\alpha, r_{n-1})) \leq 1 .$$

We will now prove the following lemma which we shall need later.

Lemma (A1.1): If $h(y)$ is continuous, convex, bounded in absolute value by a polynomial, and $h(0) \leq 0$, then:

$$g(x) = \int_0^\infty h(y) f(y|x) dy ,$$

is also continuous, convex, bounded in absolute value by a polynomial, and must belong to one of the following three categories:

(I) $g(x) \geq 0$ for all $x \geq 0$,

(II) $g(x) < 0$ for all $x \geq 0$ except with a possible zero at $x = 0$,

(III) there exists a unique x^* , $0 < x^* < \infty$, such that $g(x) > 0$ for all $x > x^*$ and $g(x) < 0$ for $x < x^*$ except for a possible zero at $x = 0$.

Proof: That $g(x)$ is continuous can be verified by examining $f(y|x)$ of Eq. (4.17) directly. Since $h(0) \leq 0$, we have $S^-(h) \leq 1$, thus by Theorems (4.9) and (A1.2) $g(x)$ is convex, bounded in absolute value by a polynomial, and $S^-(g(x)) \leq 1$. If $S^-(g) = 1$, then there exists a unique x^* where the sign changes, furthermore, $S^-(h)$ must be equal to 1. $h(0) \leq 0$ implies that the sign of $h(x)$ is first negative then becomes positive as x goes from 0 to ∞ . By the second half of Theorem (A1.1) this holds for $g(x)$ also, thus $g(x)$ is in the third category. If $S^-(g) = 0$, then either $g(x) \geq 0$ for all $x \geq 0$ which is in category I or $g(x) \leq 0$ for all $x \leq 0$ and with strict inequality for some x ; by convexity, $g(x)$ is in category II. ||

We will now resume the proof of the theorem. Since $G_{n-1}(\alpha, r)$ satisfies all the requirements in the above lemma, thus $\Delta_{n-1}(r_{n-2})$ belongs to one of the three categories defined in the lemma. Therefore we can choose for each category, r_{n-2}^* where: (I) $r_{n-2}^* = 0$; (II) $r_{n-2}^* = \infty$; and (III) $r_{n-2}^* = x^*$. It is clear that this decision rule is optimal because for all $r_{n-2} \geq 0$:

$$\Delta_{n-1}(\alpha, r_{n-2}) \geq 0 \quad (\text{or } \leq 0) \quad \text{if } r_{n-2} \geq r_{n-2}^* \quad (\text{or } < r_{n-2}^*).$$

At state $n-2$, we have:

$$\begin{aligned} \Delta_{n-2}(\alpha) &= \int_0^\infty [\beta_{n-2} w(r_{n-2}) - \alpha r_{n-2} + (\Delta_{n-1}(\alpha))^+ + e_{n-2}(\alpha) \\ &\quad - e_{n-1}(\alpha)] f(r_{n-2} | r_{n-3}) d r_{n-2}, \end{aligned}$$

and:

$$\begin{aligned} & \Delta_{n-2}(\alpha, r) - \Delta_{n-1}(\alpha, r) \\ &= \int_0^\infty [(\beta_{n-2} - \beta_{n-1}) w(t) + (\Delta_{n-1}(\alpha, t))^+ + e_{n-2}(\alpha) - 2e_{n-1}(\alpha) \\ &\quad + e_n(\alpha)] f(t|r) dt. \end{aligned}$$

With the assumption that $\beta_{n-2} \geq \beta_{n-1}$, $e_{n-2}(\alpha) - 2e_{n-1}(\alpha) + e_n(\alpha) \geq 0$,

then $\Delta_{n-2}(\alpha, r) - \Delta_{n-1}(\alpha, r) \geq 0$ for all r .

Similarly, for $1 \leq j \leq n-3$:

$$\begin{aligned} \Delta_j(\alpha) - \Delta_{j+1}(\alpha) &= \int_0^\infty [(\beta_j - \beta_{j+1}) w(t) + (\Delta_{j+1}(\alpha))^+ \\ &\quad - (\Delta_{j+2}(\alpha))^+ + e_j(\alpha) - 2e_{j+1}(\alpha) + e_{j+2}(\alpha)] f(t|r) dt, \end{aligned}$$

therefore

$$\Delta_j(\alpha, r) \geq \Delta_{j+1}(\alpha, r) \quad \text{for all } r \geq 0 \quad \text{if}$$

$\Delta_{j+1}(\alpha, r) \geq \Delta_{j+2}(\alpha, r)$ for all $r \geq 0$. By mathematical induction we have proved the following lemma.

Lemma (A1.2): Under the assumptions of Theorem (A1.3)

$$\Delta_j(\alpha, r) \geq \Delta_{j+1}(\alpha, r) \text{ for all } r \geq 0 \text{ and for } 1 \leq j \leq n-2.$$

The relation between $\Delta_j(\alpha)$ and $\Delta_{j+1}(\alpha)$ is summarized in the following lemma:

Lemma (A1.3): If $\Delta_{j+1}(\alpha, r_j)$ possesses all the properties of $g(x)$ in Lemma (A1.1), then so does $\Delta_j(\alpha, r_{j-1})$.

Proof: By definition:

$$\Delta_j(\alpha, r) = \int_0^\infty [\beta_j w(t) - \alpha t + (\Delta_{j+1}(\alpha, t))^+ + e_j(\alpha) - e_{j+1}(\alpha)] f(t|r) dt.$$

(A1.2)

Since $\Delta_{j+1}(\alpha, t)$ is assumed to be convex, so are $(\Delta_{j+1}(\alpha, t))^+$ and the integrand of Eq. (A1.2), thus $\Delta_j(\alpha, r)$ is convex. In addition, $\Delta_j(\alpha, r)$ is continuous, bounded in absolute value by a polynomial.

If $\Delta_{j+1}(\alpha)$ is in category I, i.e., $\Delta_{j+1}(\alpha) \geq 0$, then by Lemma (A1.2) $\Delta_j(\alpha) \geq \Delta_{j+1}(\alpha) \geq 0$ is also in this category. If $\Delta_{j+1}(\alpha)$ is in category II or III, then $\Delta_{j+1}(\alpha, 0) \leq 0$. Therefore the integrand of Eq. (A1.2) at $t = 0$ is equal to:

$$e_j(\alpha) - e_{j+1}(\alpha) \leq 0 ,$$

by assumption (ii), thus the integrand satisfies all the requirements on $h(y)$ in Lemma (A1.1). By the same lemma, $\Delta_j(\alpha, r_{j-1})$ has all the properties of $g(x)$. ||

Since $\Delta_{n-1}(\alpha)$ possesses all the properties of $g(x)$ in Lemma (A1.1), $\Delta_j(\alpha)$ also has all these properties by the above lemma. We choose r_j^* according to the following rule: If $\Delta_j(\alpha)$ is in category I then $r_{j-1}^* = 0$. If $\Delta_j(\alpha)$ is in category II then $r_{j-1}^* = \infty$. If $\Delta_j(\alpha)$ is in category III then r_{j-1}^* is chosen as the unique zero-crossing point where $\Delta_j(\alpha, r) > 0$ for $r > r_{j-1}^*$. It is obvious that the decision rule based on r_{j-1}^* is optimal at state j ; thus r_j^* , for $0 \leq j \leq n-2$, gives us a decision rule which is optimal. By Lemma (A1.2), if $r_j^* = 0$ (i.e., category I) then $r_{j-1}^* = 0$; if $0 < r_j^* < \infty$ (i.e., category III) then by the fact that $\Delta_j(\alpha, r) \geq \Delta_{j+1}(\alpha, r) > 0$ for all $r > r_j^*$ we have $r_{j-1}^* \leq r_j^*$. If $r_j^* = \infty$, then obviously $r_{j-1}^* \leq r_j^*$. This completes the proof of the theorem. ||

Corollary: If $E(p_i) = E(p_n)$ and $E(d_2) = E(d_n)$ for $1 \leq i \leq n$, then

$e_i(\alpha) = e_n(\alpha)$ for $1 \leq i \leq n$ and the theorem holds.

Remark 1: Theorem (A1.3) holds for the special cases discussed in

Chapter 4 with constant replacement cost and time throughout all the different states.

Remark 2: In the proof of Theorem (A1.3) we have relied heavily on the stationarity of the probability transition function and assumption (i) and (iii) to keep the number of sign changes of the decision functions not greater than one. However, if the conditional density is allowed to be nonstationary and the only requirement is that $c(t)$ is non-decreasing, discarding assumptions (i) and (iii), then $\Delta_j(\alpha)$ is still convex but may have two nontrivial zero-crossing points.

APPENDIX 2

COMPUTATIONAL ASPECTS OF LINEAR REWARD RATE CASE

In our computation of integrals, the interval for "Simpson's Rule" is 0.1. The lowest standard deviation we have considered is 0.5. Therefore, we have a good approximation to the integral by the "Simpson's Rule."

The memory requirements are low. The minimum 86k memory is enough. But the integrations are very time consuming, because there are a few nested DO-loops that we have to carry out. For each cycle in a nested DO-loop, the main program has to call the Bessel function subroutine. This subroutine takes about one millisecond before it returns to the main program. Therefore, the computation is very costly even for a fixed value of α . We cannot put another DO-loop for α .

The method of search for α^* is to guess an initial value for α^* (say α_1) , then the second value for α^* (say α_2) will depend on the sign of $L_0(\alpha_1)$.

Hence we have $L_0(\alpha_1)$ and $L_0(\alpha_2)$, we can use linear interpolation to get a good approximation of α^* . The following is a FORTRAN program for our problem, where $\eta = 5$, $\beta_0 = 5$, $\beta_1 = 4$, $\beta_2 = 3$, $\beta_3 = 2$, $\beta_4 = 1$, $\beta_5 = 0$,

$$C_i(+) = 2\beta_i t, \quad 0 < t < 1.$$

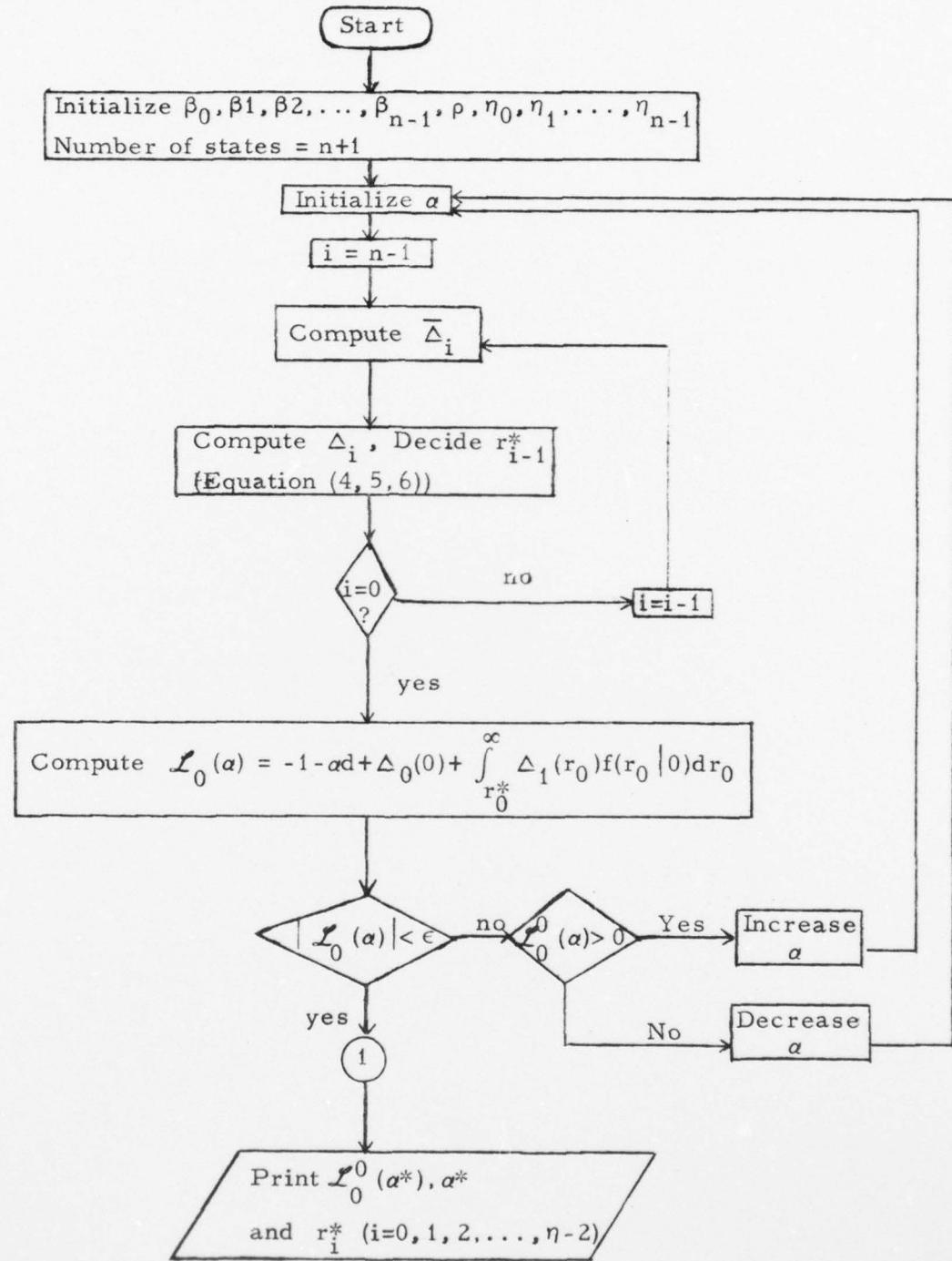
```
DIMENSION D4(200), D3(200), D2(200), D1(200)
DIMENSION Z4(200), Z3,(200), Z2(200), Z1(200)
DIMENSION V(200), Y(200)
D=1.0
C THE REPLACEMENT DURATION
P = 5.0
C THE REPLACEMENT COST
R = 0. NNNN
C THE CORRELATION ROO
F = NN. NNNNNN
C THE BRENDER'S PARAMETER "ALPHA"
A = 1. / (2. -2.*R)
B = A
C = R*A
E = SQRT(R)/(1.-R)
```

C THE PARAMETERS FOR THE CONDITIONAL DISTRIBUTION
G=2.0*F*(1.0-R)
H=8.0*(1.0-R)**2
Q=8.0*R*(1.0-R)
T=R*F
S=R**2
C THE COEFFICIENTS OF REWARD DECISION FUNCTIONS
R3S=(-(Q-T)+SQRT((Q-T)**2-4.0*S*(H-G)))/(2.0*S)
KK3=(10.0)*R3S-1.0
C FIRST ZERO CROSSING
DO 15 I=1, 200
V(I)=0.1*(I-1)
DO 5 K=1, 200
D4(K)=0.01*S*(K+KK3)**2+(Q-T)*(K+KK3)*0.1+H-G
W4=A*((K+KK3)*0.1)
X4=E*SQRT(0.01*(I-1)*(K+KK3))
CALL BESI(X4, 0, BI, IER)
Y(K)=D4(K)*A*EXP(-W4)*BI
5 CONTINUE
SIMP=0.0
DO 10 M=1, 197, 2
SIMP=SIMP+(0.1/3)*(Y(M)+4.*Y(M+1)+Y(M+2))
10 CONTINUE
Z3(I)=SIMP*EXP(-((I-1)*C*0.1))
D3(I)=0.02*S*(I-1)**2+(2.0*Q-T)*(I-1)*0.1+2.0*H-G+Z3(I)
15 CONTINUE
DO 16 I=1, 200
IF (D3(I)) 16, 17, 17
17 KK2=I
C SECOND ZERO CROSSING
I=200
16 CONTINUE
DO 35 I=1, 200
DO 25 K=KK2, 200
W3=A*K*0.1
X3=E*SQRT(0.01*(I-1)*K)
CALL BESI(X3, 0, BI, IER)
Y(K)=D3(K)*EXP(-W3)*BI
25 CONTINUE
SIMP=0.0
DO 24 M=KK2, 197, 2
SIMP=SIMP+(0.1/3)*(Y(M)+4.*Y(M+1)+Y(M+2))
24 CONTINUE
Z2(I)=SIMP*EXP(-((I-1)*C*9.1))*A
D2(I)=0.03*S*(I-1)**2+(3.0*Q-T)*(I-1)*0.1+3.0*H-G+Z2(I)
35 CONTINUE
DO 36 I=1, 200
IF (D2(I)) 36, 37, 37
37 KK1=I
I=200
36 CONTINUE

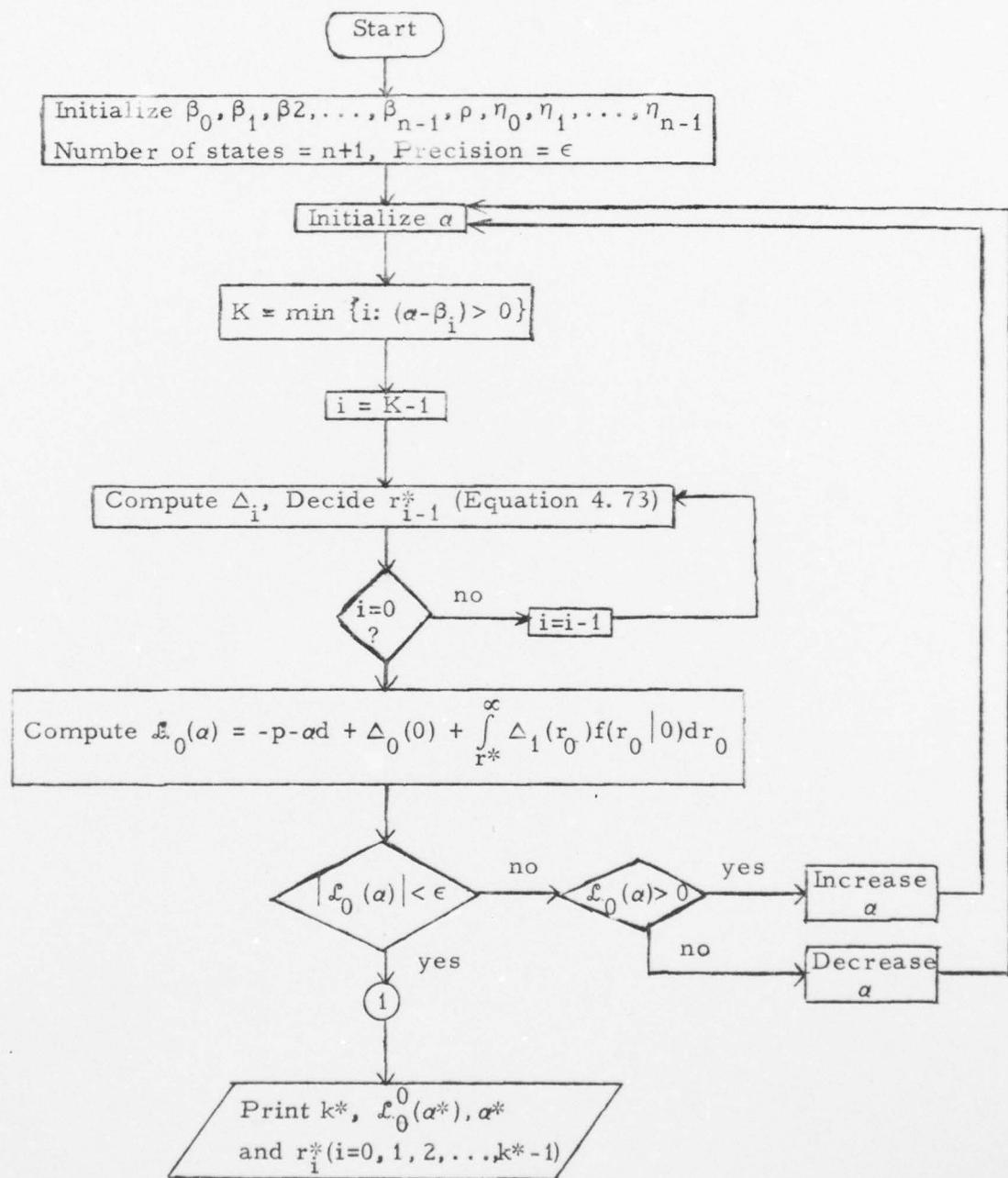
```
DO 55 I=1, 200
DO 45 K=KK1, 200
W2=A*K*0. 1
X2=E*SQRT(0. 01*(I-1)*K)
CALL BESI(X2, 0, BI, IER)
Y(K)=D2(K)*EXP(-W2)*BI
45 CONTINUE
SIMP=0. 0
DO 44 M=KK1, 197, 2
SIMP=SIMP+(0. 1/3)*(Y(M)+4.*Y(M+1)+Y(M+2))
44 CONTINUE
Z1(I)=SIMP*EXP(-((I-1)*C*0. 1))*A
D1(I)=0. 04*S*(I-1)**2+(4. 0*Q-T)*(I-1)*0. 1+4. 0*H-G+Z1(I)
55 CONTINUE
DO 56 I=1, 200
IF (D1(I)) 56, 57, 57
57 KKZ=I
I=200
56 CONTINUE
DO 65 K=KKZ, 200
W1=0. 05*K
Y(K)=D1(K)*EXP(-W1)
65 CONTINUE
SIMP=0. 0
DO 64 M=KKZ, 197, 2
SIMP=SIMP+(0. 1/3)*(Y(M)+4.*Y(M+1)+Y(M+2))
64 CONTINUE
ZZ=SIMP*0. 5
RW=-P-F*D+40-2. 0*F+ZZ
WRITE (6, 101)(RW, KK+, KK2, KK1, KKZ)
101 FORMAT (5X, F10. 6, 5X, I4, 5X, I4, 5X, I4, 5X, I4)
WRITE(6, 111)(V(I), Z3(I), D3(I), Z2(I), D2(I), Z1(I), D1(I), I=1, 200)
111 FORMAT(5X, F4. 1, 3X, F10. 4, 3X, F9. 4, 3X, F9. 4, 3X, F9. 4, 3X, F9. 4)
STOP
END
```

APPENDIX 3

FLOW CHART OF LINEAR REWARD RATE CASE



APPENDIX 4
FLOW CHART OF CONSTANT REWARD RATE
AFTER SET-UP TIME CASE



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